Ontario Mathematics Competition

(Part I)

Multiple Choice

Contestant Information

First name

Last name

Grade

General Instructions:

1. DO NOT open the contest booklet until instructed by your proctor.
2. Before the contest begins, make sure to fill in the contestant information section legibly.
3. You may use scratch paper, a ruler, a compass, and a protractor for rough work.
4. Calculators are permitted as long as they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software. Graphing calculators (GDCs) are NOT allowed.
5. Diagrams are not drawn to scale. They are intended as aids only.

Exam Format:

1. The first part of the OMC consists of twenty-five multiple choice questions to be completed in 60 minutes.
2. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct. Once you have made your choice, fill in the corresponding circle in the bubble sheet.
3. Scoring: Each correct answer is worth 5 marks.
   There is no penalty for an incorrect answer.
   Each unanswered question is worth 2 marks, to a maximum of 10 unanswered questions.
   For tiebreaks, a tiebreaker score will be calculated where a question is worth the same number of marks as its question number, e.g. question 1 is worth 1 mark.

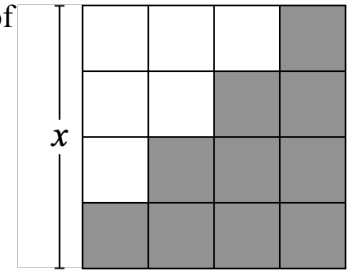
1. What is the value of $\frac{1}{21}+\frac{2}{21}+\frac{4}{21}+\frac{8}{21}+\frac{16}{21}+\frac{32}{21}$?

   (A) 3 (B) $\frac{64}{21}$ (C) π (D) $\frac{22}{7}$ (E) 4

2. If five-fourths of four-thirds of a number is 25, what is the original number?

   (A) 15 (B) 16 (C) $\frac{50}{3}$ (D) 20 (E) $\frac{125}{6}$

3. A square is divided into 16 smaller squares, as shown. If the area of the shaded region can be expressed as $a\,x^2$, find a.

   (A) $\frac{3}{8}$ (B) $\frac{2}{5}$ (C) $\frac{1}{2}$
   (D) $\frac{3}{5}$ (E) $\frac{5}{8}$

4. How many positive integers less than or equal to 100 are divisible by 2 but not divisible by 5?

   (A) 10 (B) 20 (C) 25 (D) 40 (E) 50

5. A circle with centre $(-3,\ 2)$ passes through the points $(0,\ -2)$ and $(0,\ b)$. If b is positive, what is the value of b?

   (A) 5 (B) 6 (C) 2π (D) 7 (E) 3π

6. Abby, Benjamin and Claire are each assigned a hat from a bag which contains two red hats and two blue hats. Each person cannot see their own hat, but can see the hats of the others.

   - Abby says, "I cannot determine the colour of my hat."
   - Benjamin then says, "I could not determine the colour of my hat, but after hearing you say that, I now know the colour of my hat."

   How many ways are there to arrange the hats such that this is the case?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

7. What is the value of x^2+y^2, if $x+y=20$ and $xy=30$?

   (A) 300 (B) 340 (C) 360 (D) 370 (E) 400

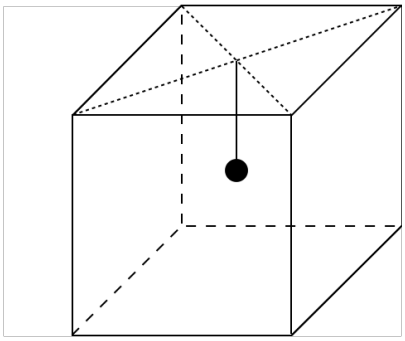
8. What is the area enclosed by $y=-6x+8$, $y=-\frac{1}{2}x+\frac{5}{2}$, the x-axis, and the y-axis?

   (A) 2 (B) $\frac{5}{2}$ (C) $\frac{31}{12}$ (D) 3 (E) $\frac{16}{3}$

9. If a number has eight divisors in total including one and itself, and two of those divisors are 21 and 51, what is the sum of all eight divisors?

   (A) 218 (B) 459 (C) 460 (D) 575 (E) 576

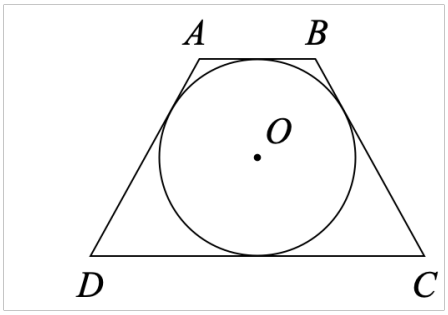
10. In the diagram, the cube has side length 11. A point is suspended from the centre of the upper surface so that it is 6 away from the bottom surface. What is the distance between the point and the closest vertex on the cube?



(A) $5\sqrt{2}$ (B) 10 (C) $\frac{3\sqrt{38}}{2}$

(D) $2\sqrt{26}$ (E) $7\sqrt{2}$

11. In an isosceles trapezoid $ABCD$, $AD = BC$. A circle with centre O and radius 4 inscribes $ABCD$. What is the length of BC if the area of $ABCD$ is 120?



(A) 12 (B) 13 (C) 14

(D) 15 (E) 16

12. 7 lilypads are arranged in a row, and numbered 1 to 7 in order. Frogbert starts at lilypad 1, and his friend Toady starts at lilypad 7. Every turn, Frogbert moves either 1 or 2 lilypads to the right, and Toady moves either 1 or 2 lilypads to the left, with equal probability. What is the probability that they will meet on the same lilypad after some number of turns?

(A) $\frac{3}{16}$ (B) $\frac{3}{8}$ (C) $\frac{25}{64}$ (D) $\frac{19}{32}$ (E) $\frac{3}{4}$

13. Triangle ABC is right angled at A. The angle bisector of A intersects BC at point D. If $AB = 1$ and $AC = 3$, what is AD?

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{3}{4}$ (C) $\frac{3\sqrt{2}}{4}$ (D) $\frac{3\sqrt{2}}{2}$ (E) $3\sqrt{2}$

14. The sum of five numbers a, b, c, d, and e taken in pairs are 67, 68, 72, 73, 77, 78, 79, 84, 85, and 89. If $a < b < c < d < e$, what is the value of d?

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

15. Bocchi is playing a game where she traces a path through the board of letters, as shown on the right. to form a word. If she can step vertically, horizontally, or diagonally to any adjacent letter, how many ways can she spell ONTARIO?

O	O	O	O	O	O	O	
O	N	N	N	I	I	O	
O	N	T	T	R	I	O	
O	N	T	A	R	I	O	
O	N	T	T	R	I	O	
O	N	N	N	I	I	O	
O	O	O	O	O	O	O	

(A) 1925 (B) 2035 (C) 2090

(D) 2145 (E) 2200

16. If $3x^2 - qx + r = 0$, where q and r are prime numbers, has distinct rational roots, what is the product of all the possible values of q?

(A) 14 (B) 15 (C) 21 (D) 33 (E) 35

17. If $\sin\theta + \cos\theta = \frac{\sqrt{2}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is the value of $\tan\theta - \cot\theta$?

(A) $-\frac{\sqrt{3}}{2}$ (B) $-\frac{8\sqrt{6}}{23}$ (C) $-\frac{\sqrt{6}}{3}$ (D) $\frac{2}{25}$ (E) $\frac{8\sqrt{6}}{23}$

18. A sequence of integers $a_1, a_2, \ldots, a_{2023}$ satisfies $a_n = 3a_{n-1} - 2a_{n-2}$ for all $n \geq 3$, with $a_1 = 7$ and $a_2 = 13$. Find the remainder of $a_1 + a_2 + \ldots + a_{2023}$ when divided by 5.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

19. The function $f(x)$ is defined over the positive integers, and $f(1) = 0$. If x is divisible by 3, then $f(x) = f\left(\frac{x}{3}\right) + 3$. Otherwise, $f(x) = 3\left(f(x-1)\right)$. Find the value of $\frac{f\left(3^{2025}-1\right)}{f\left(3^{1013}-1\right)}$.

(A) $3^{2023} - 1$ (B) $3^{2023} + 1$ (C) $3^{2023} + 4$ (D) $3^{2024} - 1$ (E) $3^{2024} + 1$

20. For any positive integer n, let $\omega(n)$ denote the number of non-negative integers m where $2^m < n$ such that $\left\lfloor \frac{n}{2^m} \right\rfloor$ is even. Find $\omega(1) + \omega(2) + \omega(3) + \cdots + \omega(1023)$. ($\lfloor x \rfloor$ is the largest integer less than or equal to x)

(A) 2023 (B) 3072 (C) 4097 (D) 8193 (E) 10240

21. In a store, there is a collection of strange bills. There is a red and a green bill of value $\$2^i$, for all natural numbers i. There is only one \$1 bill. The ***n-score*** is defined to be the number of ways to form $\$n$ using the collection of strange bills. An ***n-score*** is said to be ***tricky*** if it is divisible by 2023. Find the number of positive integers n between 1 and 5000 for which the ***n-score*** is ***tricky***.

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

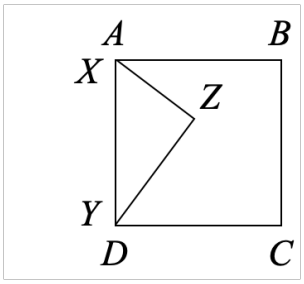
22. Let S be the sum of the number of elements in $A \cup B$ for all unordered pairs of distinct subsets A and B of $\{1, 2, 3, \ldots, 2023\}$. Find the remainder when S is divided by 5. ($A \cup B$ denotes the set of elements which are in either or both of A and B)

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

23. A binary string consisting of only "0"s and "1"s is said to be ***reflectable*** if it is a palindrome after all leading zeros and trailing zeros have been removed. For example, 011011000, 000 and 0100 are all ***reflectable***. If the number of ***reflectable*** 2023-digit binary strings can be represented as $5\left(2^a\right) - b$, where b is minimized, and a, b are both positive integers, find $a + b$. (a palindrome is a number that reads the same forwards and backwards)

(A) 4173 (B) 5064 (C) 5065 (D) 6065 (E) 6067

24. In the diagram, $ABCD$ is a square box, and XYZ is a triangle with right angle at Z and side lengths 3, 4 and 5. Initially, X and Y lay on A and B, respectively, as shown. The triangle slides counter-clockwise inside the box such that X and Y always remain on the sides of $ABCD$, i.e. side XY initially overlaps AD and slides counter-clockwise to overlap DC, etc. What is the total distance traveled by Z if XYZ moves around the square back to its original position exactly once?



(A) $2\sqrt{2}\pi$ (B) $6\sqrt{3}$ (C) 3

(D) 12 (E) $2\sqrt{5}\pi$

25. Find the sum of the digits of the square of the number formed by 2023 "1"s, i.e. $(111...11)^2$.

(A) 18180 (B) 18193 (C) 18208 (D) 18225 (E) 18226

FILLING INSTRUCTIONS:

1. Print your name and information clearly in the designated spaces before the contest period begins. DO NOT start filling until you are instructed to do so.
   - Use your full, unabbreviated name.
   - If the length of your information exceeds the number of boxes provided, write as many characters as possible starting from the left and leave the rest.
2. Do not use this sheet as scratch paper. Leave the white space untouched.
3. Manage your time wisely. Answer the questions you know how to solve first. There is only one correct answer per question.
4. Use a pencil to fill in the bubbles, do not use pens. Erase completely if you intend to correct an answer.
5. To answer a question, completely blacken the corresponding circle. A circle is blackened if the white paper beneath is not visible.

ENTER YOUR NAME IN THE BOXES BELOW
FIRST NAME
LAST NAME

EXAM DATE	
MONTH	DAY

SCHOOL YOU ATTEND		
ACRONYM	BOARD	GRADE

OFFICE USE ONLY									
SCORE									
1		6		11		16		21	
2		7		12		17		22	
3		8		13		18		23	
4		9		14		19		24	
5		10		15		20		25	
TOTAL									

Contest Answers

1	A○	B○	C○	D○	E○	11	A○	B○	C○	D○	E○	21	A○	B○	C○	D○	E○
2	A○	B○	C○	D○	E○	12	A○	B○	C○	D○	E○	22	A○	B○	C○	D○	E○
3	A○	B○	C○	D○	E○	13	A○	B○	C○	D○	E○	23	A○	B○	C○	D○	E○
4	A○	B○	C○	D○	E○	14	A○	B○	C○	D○	E○	24	A○	B○	C○	D○	E○
5	A○	B○	C○	D○	E○	15	A○	B○	C○	D○	E○	25	A○	B○	C○	D○	E○
6	A○	B○	C○	D○	E○	16	A○	B○	C○	D○	E○						
7	A○	B○	C○	D○	E○	17	A○	B○	C○	D○	E○						
8	A○	B○	C○	D○	E○	18	A○	B○	C○	D○	E○						
9	A○	B○	C○	D○	E○	19	A○	B○	C○	D○	E○						
10	A○	B○	C○	D○	E○	20	A○	B○	C○	D○	E○						