OMC

WOLFRAM

AoPS

TTmath
Top Math School (www.ttmath.ca)

Ontario Mathematics Competition

(Part II)

Long Answer

Contestant Information

First name

Last name

Grade

General Instructions:

1. DO NOT open the contest booklet until instructed by your proctor.
2. Before the contest begins, make sure to fill in the contestant information legibly.
3. Calculators are permitted as long as they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software. Graphing calculators (GDCs) are NOT allowed.
4. Answers must be expressed as simplified exact numbers.
5. You may add extra pages at the end as long as they are clearly labeled at the top right corner with your full name, grade, school, problem number, and page number.

Exam Format:

The second part of the OMC consists of four questions to be completed in 60 minutes. Questions 1 to 3 are worth 30 marks each, and question 4 is worth 35 marks.

Parts of each question may be one of two types:

1. SHORT ANSWER:
   - worth a maximum of 10 marks each.
   - full marks are awarded for a correct answer **in the specified field.**
   - partial marks may be given only if **relevant** work is shown in the space provided.
2. LONG ANSWER:
   - worth the remainder of the 30 or 35 marks for the question.
   - full marks are awarded for a correct answer and clear, complete solutions written in the appropriate location in the answer booklet.

1. (a) Find the value of x for which $\frac{5}{3} + \frac{13}{x} = 2$. [SHORT ANSWER]

   (b) Find all real values of x for which $\frac{5}{x} + \frac{7}{(x+1)} = \frac{19}{x\,(x+1)}$ where $x \neq 0$ and $x \neq -1$. [SHORT ANSWER]

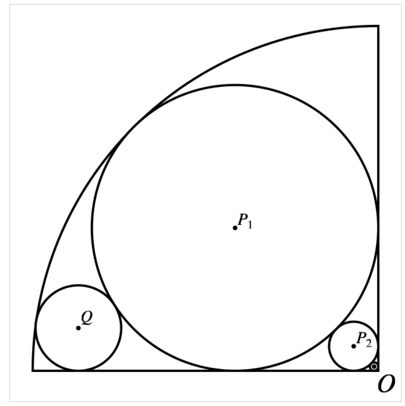
   (c) Find all real values of x for which $\frac{3}{x^2} + \frac{5}{x} = 2$. [LONG ANSWER]

Your final answer (a):
Your final answer (b):

2. Define the ***n-adjacent*** set, A_n, as the set of the first n positive integers, i.e. $A_n = \{1, 2, 3, \cdots, n-1, n\}$.
   (a) How many 9 digit integers using each element of A_9 exactly once in its digits are divisible by 4? [SHORT ANSWER]
   (b) If sets B and C each consist of 8 random elements from the set A_9, what is the probability that the difference between the sums of all elements of B and C is divisible by 4? [SHORT ANSWER]
   (c) Find the smallest positive integer value of n such that the average value of $|x-y|$ for all ordered pairs (x, y) for distinct x and y in A_n is at least 2023. [LONG ANSWER]

Your final answer (a):
Your final answer (b):

3. For each of the circles in the infinite set P, P_n is externally tangent to P_{n-1} and is tangent to the radii of the quarter circle O, as shown in the diagram. P_1 has radius 4.
   (a) Find the radius of the quarter circle O. [SHORT ANSWER]
   (b) Find the sum of the areas of all circles in the set P. [LONG ANSWER]
   (c) Draw a circle Q so that it is externally tangent to P_1, internally tangent to the arc of O, and tangent to a radius of O, as shown in the diagram. Find the radius of Q. (express your answer as $a\sqrt{b}+c$ for rational a and c and integer b) [LONG ANSWER]



Your final answer (a):

4. (a) A ***joli function*** f is a function defined over the real numbers that satisfies $f(0) = -1$ and $\left(f(x) - f(y) - \sin x + \sin y\right)\left(f(x) - f(y) - x^2 + y^2\right) = 0$ for all real x and y. Find all ***joli functions***. [LONG ANSWER]

   (b) A ***verminous function*** g is an even function defined over the real numbers that satisfies $g(g(x)) + 2y(g(x)) + g(y) = g(x^2 + y)$ and $g(xy) = g(x)g(y)$ for all real x and y. Find all ***verminous functions***. [LONG ANSWER]

   (c) Find all ordered pairs of integers $(x,\ y)$ with $|x| \leq 360$ and $|y| \leq 360$, that satisfy $g\left(f(x)\right)\left(g\left(y\right) + 3\right)\left(g\left(y\right) - 3\right) + 4f(x)\left(g\left(y\right)^2 + 108\right) + 4g\left(g\left(y\right)\right) = 5184$ for some ***joli function*** f and ***verminous function*** g. (all trigonometric functions are in degrees) [LONG ANSWER]

4. (cont'd)