OMC

Ontario Mathematics Competition

Monday, October 16, 2023 -
Friday, October 20, 2023

General Instructions:

1. **DO NOT** open the contest booklet until instructed by your proctor.
2. Before the contest begins, the proctors will give you a few minutes to read the instructions and fill in the contestant information section in your bubble sheet. There is no need to rush. Make sure to fill in the required fields legibly.
3. Diagrams are **NOT** drawn to scale. They are intended as aids only.
4. You are allowed scratch paper, a ruler, a compass, and a protractor for rough work.
5. Calculators are permitted as long as they do not have any of the following features:
   (i) internet access
   (ii) the ability to communicate with other devices
   (iii) information previously stored by students (such as formulas, programs, notes, etc.)
   (iv) a computer algebra system
   (v) dynamic geometry software. Graphing calculators (GDCs) are NOT allowed

Exam Format:

1. The OMC consists of twenty-five multiple-choice questions to be completed in 60 minutes.
2. Each question is followed by answers marked A, B, C, D, and E. There is only one correct answer for each question.
3. Scoring: Each correct answer is worth 3 marks.
   There is no penalty for an incorrect answer.
   Each unanswered question is worth 1 mark.
   For tiebreaks, a tiebreaker score will be calculated where a correct answer is worth the same number of marks as its question number.
   For example, question 1 is worth 1 mark.

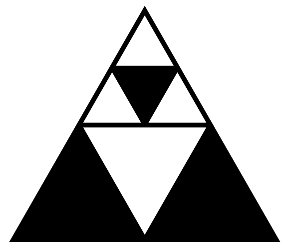
1. If $p = 7$ and $q = 2(p - 5)$, compute $q(p + 5) - 2$.

   A. 286 B. 142 C. 100 D. 61 E. 46

2. The operation § is defined by $x§y = x^2 + xy + y^2$. Find $(1§2)§3$.

   A. 49 B. 58 C. 70 D. 79 E. 91

3. In the figure shown, an equilateral triangle is split into four congruent smaller triangles, one of which is split again in the same way. What fraction of the figure is shaded?



   A. $\frac{1}{2}$ B. $\frac{13}{24}$ C. $\frac{5}{9}$ D. $\frac{9}{16}$ E. $\frac{7}{12}$

4. Compute $2021 + \frac{2023! + 2024!}{2022! \times 2025}$, where $n! = 1 \times 2 \times \ldots \times (n-1) \times n$.

   A. 4044 B. $\frac{4044}{2023}$ C. $\frac{4088484}{2023}$ D. $\frac{4094552}{2023}$ E. 2023

5. Cards are drawn one at a time from a deck containing 3 red cards and 2 black cards. What is the probability that all the black cards are drawn before all the red cards are drawn?

   A. $\frac{2}{5}$ B. $\frac{1}{10}$ C. $\frac{7}{10}$ D. $\frac{3}{5}$ E. $\frac{2}{3}$

6. Positive numbers x, y, a, b satisfy the inequalities $x < y$ and $a < b$. How many of the following inequalities are necessarily true?

   (i) $x + a - b < y - a + b$

   (ii) $\frac{x+y}{b} < \frac{x+y}{a}$

   (iii) $\frac{x}{b} < \frac{y}{a}$

   (iv) $a - y < b - x$

   (v) $ax < by$

   A. 1 B. 2 C. 3 D. 4 E. 5

7. The perpendicular bisector of a radius of a circle intersects the circle at points A and B. Given that $AB = 12$, find the radius of the circle.

   A. $2\sqrt{3}$ B. 6 C. $4\sqrt{3}$ D. 8 E. $6\sqrt{2}$

8. How many possible values of a are there such that $x^2 + ax + 36$ divides the product $x(x+1)...(x+35)(x+36)$?

   A. 3 B. 4 C. 5 D. 6 E. 12

9. Daniel has x dollars and Elaine has y dollars. Elaine gives Daniel a **fifth** of her money, which gets added to his bank account. Daniel then gives Elaine a **third** of his total money, which gets added to her bank account. Finally, Elaine gives Daniel a **quarter** of her money, after which she notices that they both have the same amount of money. Find the ratio $\frac{x}{y}$.

   A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{3}{5}$ D. $\frac{4}{7}$ E. $\frac{5}{7}$

10. A game of whack-a-mole consists of one mole and eight holes. On each turn, the player chooses a hole to whack randomly and the mole chooses a hole to pop out of randomly. If the player plays for seventeen turns, then $\frac{m}{n}$ is the average number of times they will hit the mole, where m, n are co-prime integers. What is $m + 10n$?

    A. 87 B. 178 C. 12 D. 78 E. 97

11. A function is defined recursively with $f(0) = 1$ and $f(x) = (2x - 1)f(x - 1)$. Find the smallest integer n such that $f(n)$ is divisible by 2023^2.

    A. 49 B. 60 C. 73 D. 91 E. 119

12. A sequence is defined with $a_1 = 2, a_2 = 3$, and $a_n = a_{n-1} \cdot a_{n-2} - 1$ for $n > 2$. Find the last digit of a_{2023}.

    A. 7 B. 9 C. 4 D. 2 E. 5

13. Evaluate

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{2025}\right).$$

    A. $\frac{23}{45}$ B. $\frac{4}{2025}$ C. $\frac{1}{2}$ D. $\frac{1013}{2025}$ E. $\frac{169}{529}$

14. If $f(x)$ is a quadratic that satisfies $f(1) = f(f(1)) = 2$, compute $f(5) - 6f(3)$.

    A. 6 B. 2 C. -2 D. -10 E. 0

15. For how many bases b, with $3 < b < 2023$, is 2023_b divisible by 7?

    A. 576 B. 578 C. 866 D. 867 E. 1156

16. Inside of square $ABCD$ with side length 3, a circle ω with radius 1 is drawn such that it is tangent to both $\overline{AB}$ and $\overline{BC}$. Point G lies on $\overline{CD}$ such that $\overline{AG}$ is tangent to ω. Find the area of triangle AGD.

    A. $\frac{27}{8}$ B. 3 C. 4 D. $\frac{7}{2}$ E. $\frac{15}{4}$

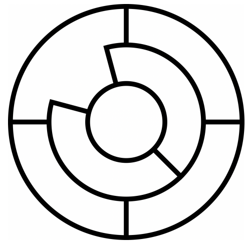
17. Avogadro, Bernoulli, Cauchy, Darwin, Einstein, Fermat, and Galileo are lining up to enter the exam room for OMC. How many ways are there for them to do so if Avogadro cannot be first in line, Cauchy cannot be 3rd in line, Einstein cannot be 5th in line, and Galileo cannot be last in line?

    A. 2790 B. 2250 C. 6 D. 2880 E. 2472

18. In triangle ABC, point D lies on $\overline{BC}$ such that $\angle ADB = 90°$. If $\overline{AD} = 5, \overline{BD} = 2$, and $\angle BAC = 45°$, find the length of $\overline{DC}$.

    A. 2 B. $\frac{15}{7}$ C. 3 D. $\frac{30}{13}$ E. $\frac{30}{17}$

19. How many ways can the shown figure be painted with six distinct colours if no two neighboring regions can share the same colour?



A. 9720 B. 12960 C. 14400 D. 15480 E. 17280

20. Player 1 and Player 2 are playing a game. On each turn, a fair coin is flipped. If it lands heads, Player 1 moves forward one square. If it lands tails, Player 2 moves forward one square. A player wins when they are 5 squares in front of the other player. If Player 1 is currently 1 square in front of Player 2, then the probability that Player 1 wins is $\frac{a}{b}$, where a, b are co-prime integers. What is $2a + 3b$?

A. 21 B. 19 C. 8 D. 17 E. 23

21. In equilateral triangle ABC, $\overline{AB} = 6$. Consider a point P which satisfies $\overline{AP} = 3$. Let Q lie on PC such that $CQ : QP = 1 : 2$. The minimum possible length of $\overline{BQ}$ can be written in the form $a\sqrt{b} + c$, where a, b, c are integers and b is not divisible by the square of any prime. Find $a + b + c$.

A. 11 B. 12 C. 8 D. 10 E. 14

22. The largest real solution to $x^4 + 2x^3 - 1686x^2 - 6x + 9 = 0$ can be expressed as $a + \sqrt{b}$, where b is a positive integer. What is $10a + b$?

A. 228 B. 234 C. 597 D. 603 E. 2012

23. How many ways are there to completely tile a 3×10 grid with 1×2 rectangles?

A. 418 B. 243 C. 162 D. 3888 E. 571

24. Parallelogram $ABCD$ has $\overline{AB} = 17$ and $\overline{BC} = 10$. Two circles with diameters AB and BC respectively are drawn, which intersect each other at points B and P with $\overline{BP} = 8$. If the two diagonals of $ABCD$ have lengths x and y, find $|x^2 - y^2|$.

A. 266 B. 538 C. 104 D. 310 E. 907

25. The minimum value of

$$\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 + 14x + 49} + \sqrt{x^2 + y^2 - 16y + 64},$$

where x and y are real numbers, is z. If $z^2 = a + b\sqrt{c}$, where c is not divisible by the square of any prime, find $a + b + c$.

A. 200 B. 128 C. 256 D. 131 E. 172

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You may use this page for rough work. Only answers filled in the bubblesheet will be evaluated.
Partial marks are not awarded for work shown.

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Jane Street Maplesoft TTmath WOLFRAM

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ID LABEL	Proctors Notes:
Name: ______________________ Grade: _________	______________________

Contestant ID						
	School ID			Test ID		
O						
	0	0	0	0	0	0
	1	1	1	1	1	1
	2	2	2	2	2	2
	3	3	3	3	3	3
	4	4	4	4	4	4
	5	5	5	5	5	5
	6	6	6	6	6	6
	7	7	7	7	7	7
	8	8	8	8	8	8
	9	9	9	9	9	9

FILLING INSTRUCTIONS:

1. Place your Contestant ID Label into the shaded box on the top left of this page. A correctly pasted label should cover all the text within the box.
   - If you are not an official contestant and don't have a Contestant ID Label, please print your name* and information clearly into the designated spaces in the shaded box.
2. Clearly copy your Contestant ID into the designated spaces.
3. Do not use the margins as scratch paper.
4. Use a pencil to fill in the bubbles. DO NOT use pens. Erase completely if you intend to correct an answer.
5. To answer a question, completely blacken the corresponding circle. A circle is blackened if the white paper beneath is not visible.

*If the length of your name exceeds the space provided, write as many characters as possible, starting from the left and leaving the rest.

Shading Methods

Incorrect Methods of Shading:

Correct Methods of Shading:

Contestant's Notes:

(Partial marks are not awarded for work shown)

Contest Answers

1	A	B	C	D	E	11	A	B	C	D	E	21	A	B	C	D	E
2	A	B	C	D	E	12	A	B	C	D	E	22	A	B	C	D	E
3	A	B	C	D	E	13	A	B	C	D	E	23	A	B	C	D	E
4	A	B	C	D	E	14	A	B	C	D	E	24	A	B	C	D	E
5	A	B	C	D	E	15	A	B	C	D	E	25	A	B	C	D	E
6	A	B	C	D	E	16	A	B	C	D	E						
7	A	B	C	D	E	17	A	B	C	D	E						
8	A	B	C	D	E	18	A	B	C	D	E						
9	A	B	C	D	E	19	A	B	C	D	E						
10	A	B	C	D	E	20	A	B	C	D	E						