Ontario Invitational Mathematics Examination

Monday, December 4, 2023 -
Friday, December 8, 2023

Contestant Information:

First name

Contestant ID

O						

Last name

General Instructions:

1. **DO NOT** open the contest booklet until instructed by your proctor.
2. Diagrams are **NOT** drawn to scale. They are intended as aids only.
3. You are allowed scratch paper, a ruler, a compass, and a protractor for rough work.
4. Calculators are not permitted during this contest.
5. Express answers as simplified exact numbers as a sum of products unless otherwise indicated. For example, $\pi + 1$ and $1 + \sqrt{2}$ are simplified exact numbers.
6. If the space we provide is not sufficient for you to present your solution, you may use additional blank sheets and label them with your name, contestant ID, and question number.
7. Do not discuss the problems or solutions from this contest until after Friday, December 8, 2023.

Exam Format:

1. The OIME consists of two parts Part A and B to be completed in 120 minutes.
   Part A
   1. This part consists of seven questions, each worth 10 marks.
   2. A correct answer is worth full marks, but partial marks may be given only if **relevant** work is shown in the space provided.

   Part B
   1. This part consists of three questions, each worth 10 marks.
   2. Full marks are awarded for a correct answer and clear, complete solutions written in the appropriate location in the answer booklet.

Part A

For the questions in part A, a correct answer will receive full marks. Part marks may be awarded if relevant work is shown in the space provided in the contest booklet.

1. In regular tetrahedron $ABCD$ with side length 1, point P lies on AB such that the shortest distance from P to line CD is $\frac{3\sqrt{2}}{5}$. Find the product of all possible values of AP.

Your final answer:

2. Find all pairs of positive integers (a, b) such that

$$\gcd(a, b) + \operatorname{lcm}(a, b) = a + b + 6$$

Your final answer:

3. The polynomial $z^6 - 7z^5 + 4$ has 6 distinct roots. What is the sum of the 6th powers of the roots?

Your final answer:

4. A unit cube has one bug standing at each vertex. Every second, each bug chooses one of the three adjacent vertices to move to, uniformly at random. After 2 seconds, what is the expected number of vertices with at least one bug standing on it?

Your final answer:

5. Let $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying

$$a_n = a_1 a_{n-1} + a_1$$

for all integers n. If there exists a value of n such that

$$\frac{a_{n-1} + 1}{a_{-n}} = -729,$$

compute the sum of all possible integer values of a_1.

Your final answer:

6. Let ABC be a triangle with $AB = 5, AC = 12$, and $BC = 13$. Points E and F are on AB and AC, respectively, such that EF is parallel to BC. Point D is on BC such that $\angle EDF = 90°$. When $(DE + DF)/(AE + AF)$ is maximized, what is AD?

Your final answer:

7. Point D lies on the circumcircle of triangle ABC on arc BC not containing A. Point E lies on line segment BD such that $DE = AC$. Given that $AD \perp CE$, $BC = 24$, $CE = 7$, and the area of $\triangle ABC$ is 56, find the area of $\triangle CDE$.

Your final answer:

Part B

For each question in Part B, your solution must be well-organized with explanation or justification. Marks are awarded for completeness and clarity. A correct solution, poorly presented, will not earn full marks.

1. In Atticus' 5th grade class, every pair of students are either friends or enemies. Additionally, they are respectful kids, so they obey the following rules:

   (a) The enemy of my enemy is my friend.

   (b) The friend of my friend is my enemy.

   What is the maximum possible number of students in the class?

1. (continued)

2. A *Goatjo* number is a positive integer a that can be written as

$$a = \frac{-2b}{b^2 - 3}$$

for some rational number b. Determine the three smallest *Goatjo* numbers.

2. (continued)

3. An *orbital sequence* of a positive integer n is a sequence of non-negative integers such that the sum of any two consecutive terms forms a unique divisor of n. For example, $1, 1, 0, 3, 6$ is a valid orbital sequence of 18, but $1, 2, 1, 8$ is not. Let $S(n)$ be the maximum value of the sum of the terms over all orbital sequence of n. Prove that there exists a value of n such that $\dfrac{S(n)}{n} > 2023$.

3. (continued)

3. (continued)

3. (continued)