OMC

Ontario Mathematics Competition

Tuesday, October 15, 2024 -
Friday, October 25, 2024

General Instructions:

1. **DO NOT** open the contest booklet until instructed by your proctor.
2. Before the contest begins, the proctors will give you a few minutes to read the instructions and fill in the contestant information section in your bubble sheet. There is no need to rush. Make sure to fill in the required fields legibly.
3. Diagrams are **NOT** drawn to scale. They are intended as aids only.
4. You are allowed scratch paper, a ruler, a compass, and a protractor for rough work.
5. Calculators are permitted as long as they do not have any of the following features:
   (i) internet access
   (ii) the ability to communicate with other devices
   (iii) information previously stored by students (such as formulas, programs, notes, etc.)
   (iv) a computer algebra system
   (v) dynamic geometry software
   graphing calculators (GDCs) are NOT allowed.

Exam Format:

1. The OMC consists of twenty-five multiple-choice questions to be completed in 60 minutes.
2. Each question is followed by answers marked A, B, C, D, and E. There is only one correct answer for each question.
3. Scoring: Each correct answer is worth 3 marks.
   There is no penalty for an incorrect answer.
   Each unanswered question is worth 1 mark.
   For tiebreaks, a tiebreaker score will be calculated where a correct answer is worth the same number of marks as its question number. For example, question 1 is worth 1 mark.

1. Compute the value of $\dfrac{2^{-1}}{(-1)^2} + \dfrac{3^2}{2^3}$.

   A. 2 B. $\dfrac{3}{2}$ C. $-\dfrac{1}{9}$ D. $-\dfrac{7}{8}$ E. $\dfrac{13}{8}$

2. If the area of square S_1 is 16 and the area of square S_2 is 25, what is the ratio between the perimeter of S_1 and the perimeter of S_2?

   A. $\dfrac{4}{5}$ B. $\dfrac{16}{25}$ C. 1 D. $\dfrac{25}{16}$ E. $\dfrac{5}{4}$

3. David solves five problems each Saturday and Sunday, and he solves six problems each weekday. During some number of consecutive days, David solved 70 problems. Which day of the week did he start solving the problems?

   A. Monday B. Wednesday C. Thursday D. Friday E. Sunday

4. Let x and y be real numbers satisfying $\dfrac{4}{x} = \dfrac{y}{3} = \dfrac{x}{y}$. What is the value of x^3?

   A. 12 B. 24 C. 36 D. 48 E. 54

5. At a business conference, there are 48 Canadians and 102 Americans. Every Canadian shook the hands of exactly three Americans, and each American either shook the hands of exactly two Canadians or did not shake hands with anyone. How many Americans did not shake hands with anyone?

   A. 5 B. 10 C. 20 D. 30 E. 80

6. Call a positive integer *prime-increasing* if its digits from left to right are strictly increasing and every pair of adjacent digits form a prime number. What is the sum of the digits of the largest prime-increasing number?

   A. 19 B. 20 C. 21 D. 22 E. 23

7. What is the sum of all positive integers n such that $n!$ ends with exactly 6 zeros?
   ($n!$ is the product of all positive integers from 1 to n)

   A. 165 B. 135 C. 140 D. 60 E. 25

8. Points A and B lie on circle ω with radius 2 such that $\overline{AB} = 2\sqrt{3}$. If point C is chosen uniformly at random on ω, what is the probability that $\angle ACB = 60^\circ$?

   A. 0 B. $\dfrac{1}{6}$ C. $\dfrac{1}{3}$ D. $\dfrac{2}{3}$ E. 1

9. Distinct positive integers a, b, c, d satisfy $a + b + c + d = 25$. What is the maximum possible value of $ab + cd - ac - bd$?

   A. 80 B. 85 C. 88 D. 90 E. 110

10. In $\triangle ABC$, let point D lie on BC such that AD bisects $\angle BAC$. Let M be the midpoint of AC. If $\overline{DM} = 5$, $\overline{AC} = 10$, $\overline{BC} = 12$, find $\overline{AD}$.

    A. $\dfrac{3}{4}$ B. $\dfrac{12}{13}$ C. 4 D. 8 E. 13

11. Let $d(n)$ denote the number of positive integer divisors of n. Find the sum of the divisors of the smallest positive integer n such that $d(d(n)) = 6$.

    A. 72 B. 96 C. 168 D. 195 E. 234

12. What is the shape formed by all possible points (x, y) satisfying the following inequality?

$$|x + y - 1| + |x - y + 1| + |x + y + 1| + |x - y - 1| \leq 4$$

    A. Four vertices of a square.
    B. A square and its inner region.
    C. Four sides of a square.
    D. Three vertices of a triangle.
    E. Eight points.

13. The string *abaabaabaaba* consists of characters a and b. A b represents either a plus sign or a minus sign, and a can be any digit from 0 to 9. How many different values could this string evaluate to? For example, one value that it can evaluate to is $0 + 02 - 98 + 20 - 9 = -85$.

    A. 315 B. 316 C. 622 D. 630 E. 631

14. Participants of a mathematics conference stay in two hotels. Participants staying in the same hotel shook hands with each other exactly once, while participants staying in different hotels did not. The organizers noticed that the total number of handshakes is coincidentally equal to the product of the number of participants in each hotel. If the total number of participants is greater than 18 but less than 36, what is the total number of participants in the conference?

    A. 20 B. 25 C. 18 D. 30 E. 33

15. Currently having no money, Bob works for 5 days. On each day, he first receives his salary of 1 gold coin, then he decides whether or not to spend some of the coins that he has saved up to that point. If the coins are indistinguishable, how many ways are there to spend the 5 coins throughout the 5 days such that he has no coins left at the end? For example, a valid way of spending is (1, 0, 2, 0, 2).

    A. 36 B. 42 C. 84 D. 120 E. 126

16. Let there be an infinite checkerboard consisting of alternating black and white 1x1 squares. Elaine has a coin of radius $r < \frac{1}{2}$. She tosses the coin onto the checkerboard at random, and wins if it does not touch any black square. The maximum value of r such that the probability of Elaine winning is at least $\frac{1}{3}$ can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, c are integers and b is square-free. Find $a + b + c$.

    A. -3 B. 5 C. 6 D. 13 E. 15

17. A function $f(x)$ satisfies $f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$. What is a possible value of $f(0)$?

    A. -2 B. -1 C. 2 D. $3 + 2\sqrt{2}$ E. 4

18. Two balls with diameters 5 and d, respectively, can be placed inside a rectangular box with dimensions $5 \times 5 \times 7$. If the maximum possible value of d can be written as $a - \sqrt{b}$ for integers a and b where b is not a perfect square, compute $a + b$.

    A. 2 B. 43 C. 59 D. 76 E. 103

19. What is the product of all possible real numbers x which satisfy the equation

$$\log_x(2) + \left(\log_2\left(\frac{x}{4}\right)\right)^3 = \log_{16}(2x^{11}) - 3?$$

A. 1 B. 2 C. 32 D. $32\sqrt{2}$ E. 64

20. Let n be the number of nonreal complex numbers z such that $|z|$ and $|z - 2024|$ are both integers less than 2024. What are the rightmost 2 digits of n?

A. 06 B. 29 C. 59 D. 76 E. 00

21. Let $a_0, a_1, \ldots$ be a sequence of numbers with $a_0 = 1$ and $a_1 = 500$. If the equation

$$(a_n)(a_{n-2})^4 = (a_{n-1})^5 - (a_{n-1}a_{n-2})^4$$

holds for all $n \geq 2$, compute a_{2024}.

A. -2024 B. 0 C. 2024 D. 2024^3 E. 2^{2024}

22. In quadrilateral $ABCD$, $AB = 7$, $BC = 5$, and $CD = 6$. Points E, F are the midpoints of AB and CD respectively, and M, N are the midpoints of AC and BD respectively. If we define $S = EM + MN + NF$, then there exists a real number X such that $S < X$ for all possible quadrilaterals $ABCD$. If the minimum possible value of X can be written as $\frac{a}{b}$ for integers a and b, what is $a + b$?

A. 6 B. 7 C. 23 D. 25 E. 27

23. If a and b are positive integers satisfying

$$7\gcd(a^2 + b, a + b^2) = \operatorname{lcm}(a, b),$$

compute the smallest possible value of $a + b$.

A. 16 B. 23 C. 28 D. 32 E. 48

24. Turbo the Snail starts in the middle cell of a 5×5 grid. If on every move he travels to an orthogonally adjacent cell (a cell that shares a side with his current cell) with uniform probability, what is the expected number of moves that he will take before arriving at the center once again?

A. 13 B. 14 C. $\frac{27}{2}$ D. 19 E. 20

25. A game is played on a regular hexagon with the following rules:

- Each vertex starts with a value of 14.
- On each move, you can decrease the value of three adjacent vertices each by 1.
- No vertex is allowed to have a value less than 0 at any point.

Let n be the number of ways to make a series of moves such that all vertices end with a value of 0. Compute the remainder when n is divided by 7.

A. 0 B. 1 C. 3 D. 5 E. 6

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You may use this page for rough work. Only answers filled in the bubblesheet will be evaluated.
Partial marks are not awarded for work shown.