OIME 2023 Official Solutions

Ontario Competitive Mathematics Committee

December 2023

Part A

For the questions in part A, a correct answer will receive full marks. Part marks may be awarded if relevant work is shown in the space provided in the contest booklet.

1. In regular tetrahedron $ABCD$ with side length 1, point P lies on AB such that the shortest distance from P to line CD is $\frac{3\sqrt{2}}{5}$. Find the product of all possible values of AP.

   proposed by: Daniel Chen

   Solution:
   Let Q be the point on CD such that PQ is perpendicular to CD. By definition, PQ is the shortest distance between P and CD. Using symmetry, Q is the midpoint of CD, and we can use the ratio of sides in a $30-60-90$ triangle to find that $AQ = BQ = \frac{\sqrt{3}}{2}$.

   Now we present two ways of finishing this problem:

   Method 1: Stewart's Theorem
   Let $AP = x$. Applying Stewart's on $\triangle ABQ$, we have

   $$x(1-x) + \left(\frac{3\sqrt{2}}{5}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 x + \left(\frac{\sqrt{3}}{2}\right)^2 (1-x)$$

   $$x^2 - x + \frac{3}{100} = 0$$

   Which has positive roots. By Vieta's theorem, the product of possible values of x is $\boxed{\frac{3}{100}}$.

   Method 2: Pythagorean Theorem
   Let the foot of the altitude from Q to AB be E, and set $PE = x$. By the Pythagorean Theorem, $QE = \frac{\sqrt{2}}{2}$. Now we have

   $$x^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{3\sqrt{2}}{5}\right)^2$$

   $$x = \pm\sqrt{\frac{11}{50}} \implies AP = \frac{1}{2} \pm \sqrt{\frac{11}{50}}$$

   From there, we can calculate the product to be $\boxed{\frac{3}{100}}$.

2. Find all pairs of positive integers (a, b) such that

$$\gcd(a, b) + \text{lcm}(a, b) = a + b + 6.$$

proposed by: Charles Ran

Solution:

Set $k = \gcd(a, b)$, and write $a = km$, $b = kn$. Note that $\gcd(m, n) = 1$ and $\text{lcm}(a, b) = \text{lcm}(km, kn) = kmn$. We substitute this into the equation and do a key factoring:

$$k + kmn = km + kn + 6$$
$$\iff k(m-1)(n-1) = 6$$

Thus, 6 must be divisible by k, so $k \in \{1, 2, 3, 6\}$. We proceed with casework on k, counting the number of positive integer solutions (m, n) for each case, each giving a unique solution (a, b).

Case I: $k = 1$

$$(m-1)(n-1) = 6$$

In this case, $(m, n) \in \{(7, 2), (4, 3), (3, 4), (2, 7)\}$. We have four solutions.

Case II: $k = 2$

$$(m-1)(n-1) = 3$$

In this case, $(m, n) \in \{(4, 2), (2, 4)\}$. However, none of these cases satisfy $\gcd(m, n) = 1$, and we have no solutions.

Case III: $k = 3$

$$(m-1)(n-1) = 2$$

In this case, $(m, n) \in \{(3, 2), (2, 3)\}$, which gives the two solutions $(a, b) \in \{(9, 6), (6, 9)\}$.

Case IV: $k = 6$

$$(m-1)(n-1) = 1$$

In this case, $(m, n) = (2, 2)$. However, $\gcd(m, n) \neq 1$ and we have no solutions.

In total, we count 6 solutions: $\boxed{(a, b) \in \{(7, 2), (4, 3), (3, 4), (2, 7), (9, 6), (6, 9)\}}$.

3. The polynomial $z^6 - 7z^5 + 4$ has 6 distinct roots. What is the sum of the 6th powers of the roots?

   proposed by: Michael Li

   Solution:
   Let the roots be $r_1, r_2, \cdots r_6$. We find an expansion for $r_1^6 + r_2^6 + \cdots r_6^6$ using Newton's Sums. Define $S_k = r_1^k + r_2^k + \cdots r_6^k$.
   By Vieta's or applying Newton's sums, we have $S_1 = 7$. We then have:

$$
\begin{aligned}
S_1 &= 7 \\
S_2 &= 7 \cdot S_1 - 2 \cdot 0 \\
S_3 &= 7 \cdot S_2 - 0 \cdot S_1 + 3 \cdot 0 \\
S_4 &= 7 \cdot S_3 - 0 \cdot S_2 + 0 \cdot S_1 - 4 \cdot 0 \\
S_5 &= 7 \cdot S_4 - 0 \cdot S_3 + 0 \cdot S_2 - 0 \cdot S_1 + 5 \cdot 0 \\
S_6 &= 7 \cdot S_5 - 0 \cdot (\cdots) - 6 \cdot 4 \\
&= 7^6 - 24 \\
&= 117625
\end{aligned}
$$

   Thus, we have $r_1^6 + r_2^6 + \cdots r_6^6 = S_6 = \boxed{117625}$.

4. A unit cube has one bug standing at each vertex. Every second, each bug chooses one of the three adjacent vertices to move to, uniformly at random. After 2 seconds, what is the expected number of vertices with at least one bug standing on it?

proposed by: Shanna Xiao

Solution:

Let P the probability that a vertex has at least one bug on it after 2 seconds. We can instead consider the probability that the vertex does not have a bug, which contains two conditions:

1. The bug previously on the vertex moves somewhere else. The bug can move anywhere on the first move, and just needs to not go back on the second move, so this is $\dfrac{2}{3}$.

2. No other bug arrives on the vertex. Consider just one of the other bugs. By parity, it must be exactly 2 moves away. Then the probability that it travels to the current vertex is $\frac{2}{9}$. There are 3 possible other bugs, so by complementary counting, the combined probability is

$$\left(1 - \frac{2}{9}\right)^3 = \frac{343}{729}.$$

Now we can finally calculate the probability P:

$$\begin{aligned} P &= 1 - \left(\frac{2}{3}\right)\left(\frac{343}{729}\right) \\ &= \frac{1501}{2187} \end{aligned}$$

Since this is the same for all 8 vertices, we apply linearity of expectation to find that the answer is $\boxed{\dfrac{12008}{2187}}$.

5. Let $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying

$$a_n = a_1 a_{n-1} + a_1$$

for all integers n. If there exists a value of n such that

$$\frac{a_{n-1}+1}{a_{-n}} = -729,$$

compute the sum of all possible integer values of a_1.

proposed by: Daniel Chen

Solution:
Assume that $a_1 \neq 0, 1$, otherwise $a_n \equiv 0$, n respectively, leading to no solution. We claim that for all integers n,

$$a_n = \frac{a_1^{n+1} - a_1}{a_1 - 1}.$$

We will prove this by induction. The base case is $a_1 = a_1$, which is obviously true. Assume that the induction hypothesis is true for a_n. Then

$$\begin{aligned} a_{n+1} &= a_1 \left(\frac{a_1^{n+1} - a_1}{a_1 - 1} + 1 \right) \\ &= \frac{a_1^{n+2} - a_1}{a_1 - 1} \end{aligned}$$

Similarly, in the other direction,

$$\begin{aligned} a_{n-1} &= \frac{a_n}{a_1} - 1 \\ &= \frac{a^{n+1} - a_1}{a_1(a_1 - 1)} - 1 \\ &= \frac{a_1^n - a_1}{a_1 - 1} \end{aligned}$$

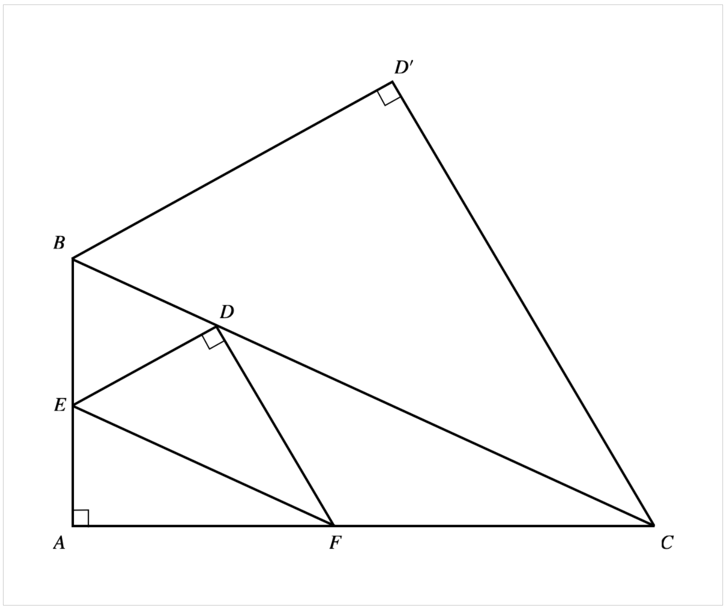
So our induction is complete. Now we can directly plug this expression into the equation:

$$\begin{aligned} -729 &= \left(\frac{a_1^n - a_1}{a_1 - 1} + 1 \right) \left(\frac{a_1 - 1}{a_1^{1-n} - a_1} \right) \\ &= -a_1^{n-1} \end{aligned}$$

Thus we just need 729 to be a perfect power of a_1 in order to have an integer solution of n. The possible values of a_1 are $\pm 3, 9, \pm 27, 729$, which sum to $\boxed{738}$.

6. Let ABC be a triangle with $AB = 5, AC = 12$, and $BC = 13$. Points E and F are on AB and AC, respectively, such that EF is parallel to BC. Point D is on BC such that $\angle EDF = 90°$. When $(DE + DF)/(AE + AF)$ is maximized, what is AD?

proposed by: Shanna Xiao



Solution 1:
Construct D' such that $BD' \parallel ED$ and $CD' \parallel FD$. $\angle BD'C = \angle EDF = 90°$, which means that $ABD'C$ is a cyclic quadrilateral with diameter BC. By similar triangles, we have the following ratio:

$$\frac{AE}{AB} = \frac{AF}{AC} = \frac{ED}{BD'} = \frac{FD}{CD'}$$

$$\frac{DE + DF}{AE + AF} = \frac{BD' + CD'}{AB + AC}$$

$AB + AC = 5 + 12 = 17$, so all we need to do is to maximize $BD' + CD'$. By Pythagorean Theorem, $(BD')^2 + (CD')^2 = BC^2 = 169$. By the Cauchy-Schwarz inequality,

$$((BD')^2 + (CD')^2)(1^2 + 1^2) \geq (BD' + CD')^2$$
$$13\sqrt{2} \geq BD' + CD'$$

Equality occurs when $BD' = CD'$. Since A, D, D', are collinear by homothety, AD must be the angle bisector of $\angle BAC$. By angle bisector theorem, $BD = \frac{65}{17}$, $CD = \frac{156}{17}$. By Stewart's, $AD = \boxed{\frac{60\sqrt{2}}{17}}$.

Solution 2:
Let $EF = x$. Then $AE = \frac{5}{13}x$, $AF = \frac{12}{13}x$. $\angle EAF + \angle EDF = 180°$ because 5-12-13 is a Pythagorean triple, so $AEDF$ is a cyclic quadrilateral. Similarly to solution 1, we can use the Cauchy-Schwarz inequality:

$$((DE)^2 + (DF)^2)(1^2 + 1^2) \geq (DE + DF)^2$$
$$x\sqrt{2} \geq DE + DF$$

The equality case gives the maximum value, which implies that $DE = DF = \frac{x\sqrt{2}}{2}$, so $\triangle DEF$ is an isosceles right triangle. To calculate x, we equate the sum of the heights of $\triangle AEF$ and $\triangle DEF$ to that of $\triangle ABC$:

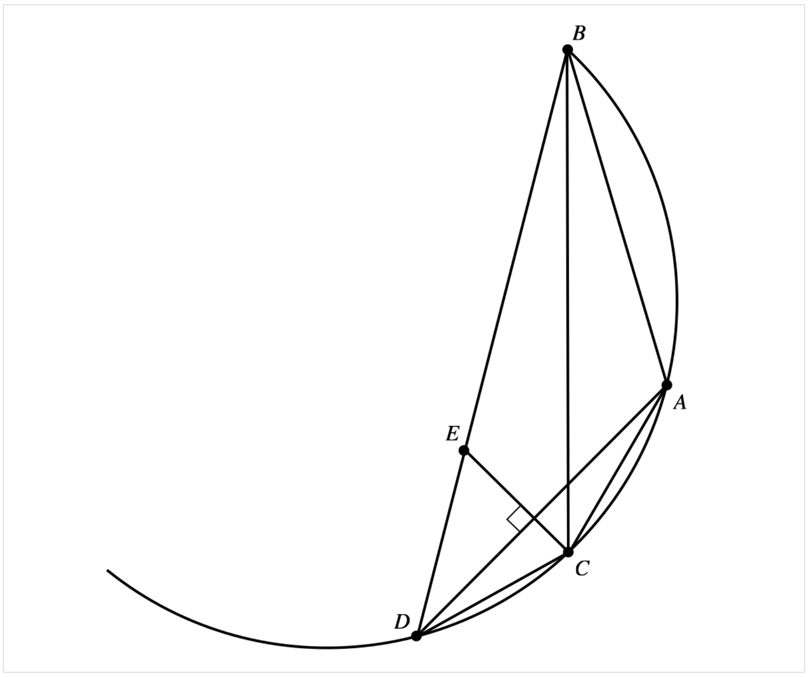
$$\frac{60}{13} \times \frac{x}{13} + \frac{x}{2} = \frac{60}{13} \implies x = \frac{1560}{289}.$$

Finally, to calculate AD, we can directly apply Ptolemy's Theorem:

$$AD \times x = \frac{x\sqrt{2}}{2}\left(\frac{5x}{13} + \frac{12x}{13}\right)$$
$$AD = \boxed{\frac{60\sqrt{2}}{17}}.$$

7. Point D lies on the circumcircle of triangle ABC on arc BC not containing A. Point E lies on line segment BD such that $DE = AC$. Given that $AD \perp CE$, $BC = 24$, $CE = 7$, and the area of $\triangle ABC$ is 56, find the area of $\triangle CDE$.

proposed by: Daniel Chen



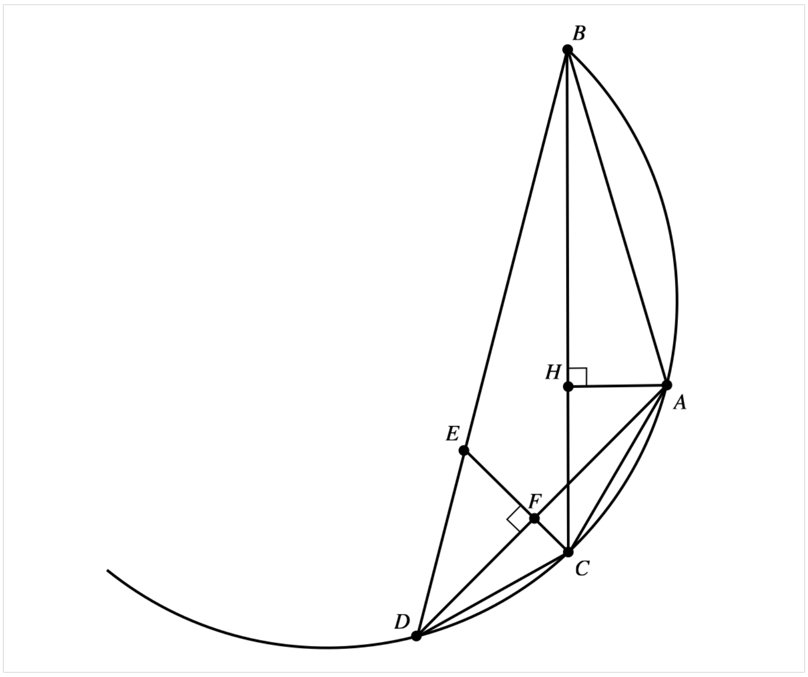
Solution 1:
Through an angle chase using the properties of cyclic quadrilaterals, we have

$$\angle DEC = 90^\circ - \angle EDA = 90^\circ - \angle BDA = 90^\circ - \angle BCA$$

$$\angle EDC = 180^\circ - \angle BAC$$

Additionally, $DE = AC$, so if we combine $\triangle ABC$ and $\triangle CDE$, we get a right triangle $\triangle BCE$ with legs BC and CE! Now the area can be directly calculated:

$$\begin{aligned}[CDE] &= [BCE] - [ABC] \\ &= 7 \times 12 - 56 \\ &= \boxed{28}.\end{aligned}$$



Solution 2 *by Sabrina Yu*:
Let H be the foot of the perpendicular from A to BC, and let F be the intersection of AD and CE. $[ABC] = BC \times AH$, so $AH = \frac{14}{3}$. Since $\angle ADB = \angle ACB$ and $AC = DE$, $\triangle AHC \cong \triangle EFD$. This implies that $CF = CE - EF = \frac{7}{3}$. Additionally, since $\angle AHB = \angle DFC = 90^\circ$ and $\angle ABH = \angle FDC$, $\triangle CFD \sim \triangle AHB$.

Set $DF = x$. Then $HC = x, BH = 24 - x$. Using similar triangle ratios on $\triangle CFD$ and $\triangle AHB$,

$$\frac{CF}{AH} = \frac{DF}{HB} \implies \frac{7}{3} \times \frac{3}{14} = \frac{x}{24 - x} \implies x = 8.$$

Finally, we can directly calculate the area:

$$\begin{aligned}[CDE] &= \frac{1}{2} DF \cdot CE \\ &= \boxed{28}.\end{aligned}$$

Part B

For the following section, a complete and correct solution is required to receive full marks. Part marks may be awarded for relevant work. Each question is worth 10 marks.

1. In Atticus' 5th grade class, every pair of students are either friends or enemies. Additionally, they are respectful kids, so they obey the following rules:

   a) The enemy of my enemy is my friend.

   b) The friend of my friend is my enemy. (they are very sensitive)

   What is the maximum possible number of students in the class?

   proposed by: Daniel Chen

   Solution:
   We will show that the maximum possible number of students is $\boxed{5}$.

   For the sake of contradiction, assume that we can have 6 or more people in the class. Note that we cannot have 3 people who are all friends or all enemies with each other, otherwise there are two people who are both friends and enemies, which is impossible. By pigeonhole, there exists three people who have the same relationship with a person (friend or enemy). Since the enemy of the enemy is a friend, and vice versa, all three of them must be friends or enemies. This is a contradiction.

   Now we show that it is possible to have 5 students. Make the 5 students stand in a circle, and have each student be friends with their two neighbors and enemies with any other student. No three people are all friends or all enemies, so this works.

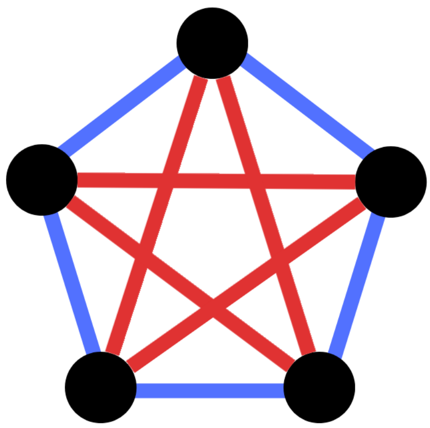


Figure 1: Blue lines are friends, red lines are enemies.

Additionally, see Ramsey's Theorem.

2. A *Goatjo* number is a positive integer a that can be written as

$$a = \frac{-2b}{b^2 - 3}$$

for some rational number b. Determine the three smallest *Goatjo* numbers.

proposed by: Charles Ran

Solution:

Rewrite the equation as a quadratic in b:

$$ab^2 + 2b - 3a = 0$$

Note that $b^2 \neq 3$. Using the quadratic formula,

$$b = \frac{-2 \pm \sqrt{4 + 12a^2}}{2a} = \frac{-1 \pm \sqrt{1 + 3a^2}}{a}$$

Thus, a positive integer a is a *Goatjo* number if and only if

$$b = \frac{-1 \pm \sqrt{1 + 3a^2}}{a} \in \mathbb{Q} \iff \sqrt{1 + 3a^2} \in \mathbb{Z}^+$$

which is true if and only if $1 + 3a^2 = c^2$ for some positive integer c. This is a Pell equation:

$$c^2 - 3a^2 = 1$$

With a minimal solution $(a_1, c_1) = (1, 2)$. Thus, all positive integer solutions $(a_i, c_i)_{i \in \mathbb{N}}$ are generated by

$$c_i + a_i\sqrt{3} = (2 + \sqrt{3})^i$$

Setting $i = 1, 2, 3$, we obtain the three smallest solutions $(a, c) \in \{(1, 2), (4, 7), (15, 26)\}$. Alternatively, we could choose to check values of a up to 15, which would be more time consuming.

The three smallest *Goatjo* numbers are $\boxed{1, 4, 15}$.

3. An *orbital sequence* of a positive integer n is a sequence of non-negative integers such that the sum of any two consecutive terms forms a unique divisor of n. For example, $1, 1, 0, 3, 6$ is a valid orbital sequence of 18, but $1, 2, 1, 8$ is not. Let $S(n)$ be the maximum value of the sum of the terms over all orbital sequence of n. Prove that there exists a value of n such that $\frac{S(n)}{n} > 2023$.

proposed by: Daniel Chen

Solution: For some positive integer n, consider the sequence

$$a_1, a_2, \ldots, a_{d(n)+1}$$

where $d(n)$ represents the number of divisors of n, and $a_i + a_{i+1} = d_i$, where d_i is the ith smallest divisor of n. Setting $a_1 = 0$, since $a_i + a_{i+1} = d_i > d_{i-1} \geq a_i$, we must have a_{i+1} positive. This is clearly an orbital sequence of n. Hence,

$$\begin{aligned} S(n) &\geq a_1 + a_2 + \ldots a_{d(n)+1} \\ &= \frac{1}{2}(\sigma(n) + a_1 + a_{d(n)+1}) \\ &\geq \frac{1}{2}(\sigma(n)) \end{aligned}$$

Where $\sigma(n)$ is the sum of the divisors of n. Now if we choose $n = k!$ for some positive integer k,

$$\begin{aligned} \frac{2S(n)}{n} &\geq \frac{\sigma(k!)}{k!} \\ &\geq \sum_{i=1}^{k} \frac{1}{i} \end{aligned}$$

which is the harmonic series, and is well-known to diverge as k tends to infinity. Hence $\frac{S(n)}{n}$ is unbounded, and there exists some n where it is greater than 2023.