Ontario Mathematics Competition 2023

(Part I: Multiple Choice)

Monday, March 20, 2023 -
Friday, March 31, 2023

Solutions

Answer Key

1. A
2. A
3. E
4. D
5. B
6. C
7. B
8. C
9. E
10. C
11. D
12. C
13. C
14. A
15. B
16. E
17. B
18. A
19. E
20. C
21. B
22. B
23. B
24. D
25. B

Solutions

Problem 1

What is the value of $\frac{1}{21} + \frac{2}{21} + \frac{4}{21} + \frac{8}{21} + \frac{16}{21} + \frac{32}{21}$?

(A) 3 **(B)** $\frac{64}{21}$ **(C)** π **(D)** $\frac{22}{7}$ **(E)** 4

Solution: Since all of the fractions have a common denominator, we can add the fractions together to get

$$\begin{aligned}\frac{1}{21} + \frac{2}{21} + \frac{4}{21} + \frac{8}{21} + \frac{16}{21} + \frac{32}{21} &= \frac{1+2+4+8+16+32}{21}\\ &= \frac{63}{21}\\ &= 3.\end{aligned}$$

Therefore, the answer is $\boxed{\textbf{(A) } 3}$.

Problem 2

If five-fourths of four-thirds of a number is 25, what is the original number?

(A) 15 **(B)** 16 **(C)** $\frac{50}{3}$ **(D)** 20 **(E)** $\frac{125}{6}$

Solution: Let the original number be x. Then,

$$\left(\frac{5}{4} \times \frac{4}{3}\right) x = 25.$$

This implies that

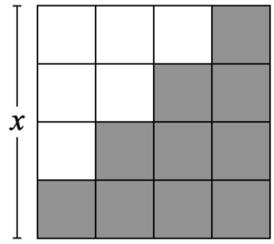
$$\begin{aligned}x &= 25 \times \frac{4}{5} \times \frac{3}{4}\\ &= \frac{25}{5} \times 3\\ &= 15.\end{aligned}$$

Therefore, the answer is $\boxed{\textbf{(A) } 15}$.

Problem 3

A square is divided into 16 smaller squares as shown. If the area of the shaded region can be expressed as ax^2, find a.

(A) $\frac{3}{8}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{5}{8}$



Solution: As seen in the figure, there are 10 shaded squares out of a total of 16 squares. Hence, the shaded region is $\frac{10}{16} = \frac{5}{8}$ of the total area.

Since the area of the square is x^2, the shaded region has area $\frac{5}{8}x^2$, which implies that $a = \frac{5}{8}$.

Therefore, the answer is $\boxed{\textbf{(E) } \frac{5}{8}}$.

Problem 4

How many positive integers less than or equal to 100 are divisible by 2, but not divisible by 5?

(A) 10 **(B)** 20 **(C)** 25 **(D)** 40 **(E)** 50

Solution: There are $\frac{100}{2} = 50$ positive integers less than or equal to 100 that are divisible by 2. But, we cannot include the numbers that are divisible by both 2 and 5. In other words, we need to subtract the number of integers that are divisible by $2 \times 5 = 10$. There are $\frac{100}{10} = 10$ positive integers less than or equal to 100 that are divisible by 10. Hence, the answer is $50 - 10 = \boxed{\textbf{(D) } 40}$.

Problem 5

A circle of centre $(-3, 2)$ passes through the points $(0, -2)$ and $(0, b)$. If b is positive, what is the value of b?

(A) 5 **(B)** 6 **(C)** 2π **(D)** 7 **(E)** 3π

Solution 1: Since the circle is symmetric about the line $y = 2$, we know that

$$\frac{b + (-2)}{2} = 2.$$

This implies that $b - 2 = 4 \implies b = \boxed{\textbf{(B) } 6}$

Solution 2: The radius of the circle is the distance between $(-3, 2)$ and $(0, -2)$, which is

$$\begin{aligned}\sqrt{(0-(-3))^2+(-2-2)^2} &= \sqrt{3^2+(-4)^2}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5.\end{aligned}$$

Using the general equation of a circle: $(x-h)^2+(y-k)^2 = r^2$, where (h, k) is the center of the circle and r is the radius of the circle, the equation of the circle is $(x+3)^2+(y-2)^2 = 25$. Plugging in $(0, b)$ into the equation gives

$$3^2 + (b-2)^2 = 25 \implies (b-2)^2 = 16.$$

Solving this equation,

$$b - 2 = \pm 4,$$

so

$$b = -2 \text{ or } 6.$$

But since $(0, -2)$ is already a point on the circle, $b \neq -2$, so $b = \boxed{\textbf{(B) } 6}$.

Problem 6

Abby, Benjamin and Claire are each assigned a hat from a bag which contains two red hats and two blue hats. Each person cannot see their own hat, but can see the hats of the others.

- Abby says "I cannot determine the colour of my hat."
- Benjamin then says, "I could not determine the colour of my hat, but after hearing you say that, I now know the colour of my hat."

How many ways are there to arrange the hats such that this is the case?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: If Abby cannot determine the colour of her hat, it must be the case that Benjamin and Claire have different coloured hats. Likewise, Abby and Claire must also have different coloured hats. Then Abby and Benjamin have the same coloured hat, and Claire has one of the other colour. There are $\boxed{\textbf{(C) } 2}$ ways to arrange the different coloured hats.

Problem 7

What is the value of $x^2 + y^2$ if $x + y = 20$ and $xy = 30$?

(A) 300 **(B)** 340 **(C)** 360 **(D)** 370 **(E)** 400

Solution 1: Since

$$x^2 + y^2 = (x + y)^2 - 2(xy) = 20^2 - 2(30) = 340,$$

the answer is $\boxed{\textbf{(B) } 340}$.

Solution 2: The second equation gives that $y = \frac{30}{x}$. Plugging this into the first equation,

$$x + \frac{30}{x} = 20.$$

Multiplying both sides by x gives

$$x^2 - 20x + 30 = 0.$$

Similarly, plugging in $x = \frac{30}{y}$,

$$y^2 - 20y + 30 = 0.$$

Adding the two equations,

$$x^2 + y^2 - 20(x+y) + 60 = 0 \implies x^2 + y^2 = 20(x+y) - 60 = 20(20) - 60 = 340.$$

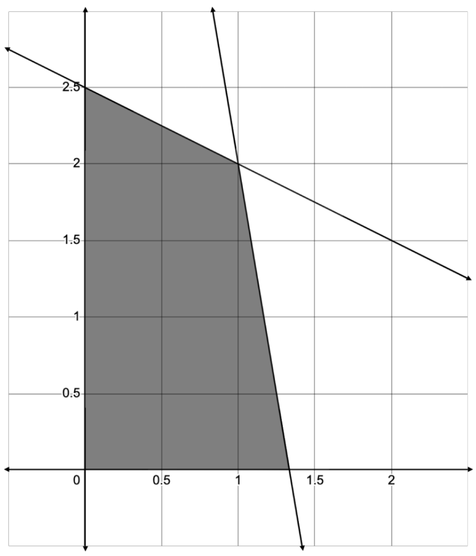
Therefore, the answer is $\boxed{\textbf{(B) } 340}$.

Problem 8

What is the area enclosed by $y = -6x + 8$, $y = -\frac{1}{2}x + \frac{5}{2}$, the x-axis and the y-axis?

(A) 2 **(B)** $\frac{5}{2}$ **(C)** $\frac{31}{12}$ **(D)** 3 **(E)** $\frac{16}{3}$

Solution: The y-intercept of the line $y = -\frac{1}{2}x + \frac{5}{2}$ is $(0, \frac{5}{2})$ and the x-intercept of the line $y = -6x + 8$ is $(\frac{4}{3}, 0)$. By equating the $y = -\frac{1}{2}x + \frac{5}{2}$ and $y = -6x + 8$, the two lines intersect at $(1, 2)$.



The area of the shaded region is $1(2) + \frac{1}{2}(0.5)(1) + \frac{1}{2}(2)(\frac{1}{3}) = \boxed{\textbf{(C)} \frac{31}{12}}$.

Problem 9

If a number has eight divisors in total including one and itself, and two of those divisors are 21 and 51, what is the sum of all eight divisors?

(A) 218 **(B)** 459 **(C)** 460 **(D)** 575 **(E)** 576

Solution: Since 21 and 51 are divisors of the number, the prime factorization of it must include 3, 7, 17. This is in fact all of the factors of the number, due to the restriction on the number of divisors. The sum of all divisors is then

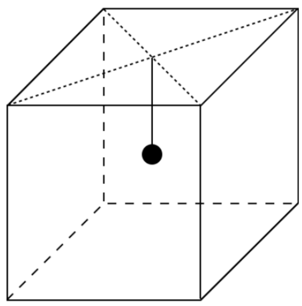
$$(1+3)(1+7)(1+17) = 576$$

Therefore the sum of all eight divisors is $\boxed{\textbf{(D)}\ 576}$.

Problem 10

In the diagram, the cube has side length 11 cm. A point is suspended from the centre of the upper surface so that it is 6 cm away from the bottom surface. What is the distance between the point and the closest vertex on the cube?

(A) $5\sqrt{2}$ **(B)** 10 **(C)** $\frac{3\sqrt{38}}{2}$ **(D)** $2\sqrt{26}$ **(E)** $7\sqrt{2}$

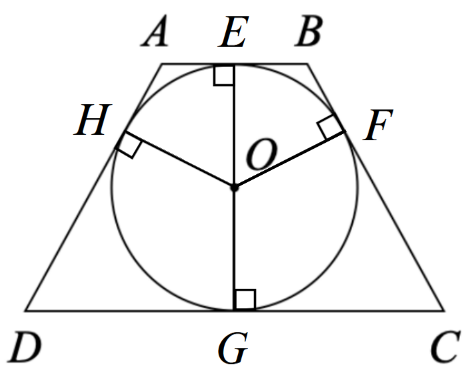


Solution: The diagonals of the upper surface of the square is equal to $\sqrt{11^2+11^2} = 11\sqrt{2}$, thus half the diagonal would be $\frac{11\sqrt{2}}{2}$ cm.
The distance from the point to the upper surface is $11 - 6 = 5$ cm. The diagonal and the suspension of the point are perpendicular, therefore by pythagorean theorem the distance from the point to any top vertex is $\sqrt{(\frac{11\sqrt{2}}{2})^2 + 5^2} = \sqrt{\frac{171}{2}}$.
After rationalizing the denominator and simplifying the roots, the distance between the point and the closest vertex is $\boxed{\textbf{(C)}\ \frac{3\sqrt{38}}{2}}$.

Problem 11

In an isosceles trapezoid $ABCD$, $\overline{AD} = \overline{BC}$. A circle with center O and radius 4 inscribes $ABCD$. What is the length of $\overline{BC}$ if the area of $ABCD$ is 120?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16



Solution: Draw perpendiculars from O to each of AB, BC, CD, and DA, as shown in the diagram.

Recall that since the circle is the incircle of $ABCD$, then $HA = AE = EB = BF$ and $HD = DG = GC = CF$. Additionally, since we have right triangles, the area of $ABCD$ is equal to $r \times sp$, where sp is the half of the perimeter of $ABCD$ and r is the inradius. Since some of the lengths are equivalent as listed above, $AH+HD+BF+FC = AD+BC$ is actually exactly half of the perimeter of $ABCD$. Then BC can be calculated as

$$\frac{120}{4} \times \frac{1}{2} = \boxed{\textbf{(D)}\ 15}.$$

Problem 12

7 lilypads are arranged in a row, and numbered 1 to 7 in order. Frogbert starts at lilypad 1, and his friend Toady starts at lilypad 7. Every turn, Frogbert moves either 1 or 2 lilypads to the right, and Toady moves either 1 or 2 lilypads left, with equal probability. What is the probability that they will meet on the same lilypad at the end of some number of turns?

(A) $\frac{3}{16}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{25}{64}$ **(D)** $\frac{19}{32}$ **(E)** $\frac{3}{4}$

Solution: Consider the distance d between the two frogs. Initially, the frogs are at a distance of 6. If after any turn $d = 0$, then they must be on the same lilypad. Every turn, the distance shrinks by 2 (probability $\frac{1}{4}$), 3 (probability $\frac{1}{2}$), or 4 (probability $\frac{1}{4}$). To reach exactly 0 from 6, the sequence of distance shrinks must be one of $(2, 2, 2), (3, 3), (4, 2)$ or $(2, 4)$. The total probability is then

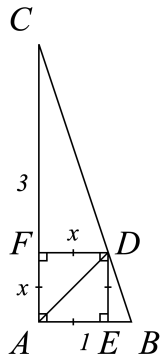
$$\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \boxed{\textbf{(C)}\ \frac{25}{64}}.$$

Problem 13

Triangle ABC is right angled at A. The bisector of A intersects $\overline{BC}$ at point D. If $AB = 1$ and $AC = 3$, what is AD?

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{3\sqrt{2}}{4}$ **(D)** $\frac{3\sqrt{2}}{2}$ **(E)** $3\sqrt{2}$

Solution: Since AD is the bisector of $\angle BAC$, and $\angle BAC = 90°$, $\angle DAC = \angle DAB = 45°$. Thus if we draw lines DE and DF perpendicular to AB and AC, the resulting quadrilateral $AEDF$ will be a square.



Now, let the distance $\overline{FD}$ be represented by x. Since $\triangle CFD \sim \triangle DEB$ (AAA), we know that

$$\frac{FD}{FC} = \frac{EB}{ED}$$
$$\frac{x}{3-x} = \frac{1-x}{x}$$
$$x = \frac{3}{4}$$

.

Since AD is a diagonal of the square $AEDF$,

$$AD = \sqrt{2}x$$
$$AD = \boxed{\textbf{(C) } \frac{3\sqrt{2}}{4}}$$

.

Problem 14

The sums of five numbers a, b, c, d, e taken in pairs are 67, 68, 72, 73, 77, 78, 79, 84, 85 and 89. If $a < b < c < d < e$, what is the value of d?

(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

Solution: Consider the sum of all of the pairwise sums of the numbers, where each number appears exactly 4 times because it can be paired with any of the other numbers:

$$4(a+b+c+d+e) = 772 \implies a+b+c+d+e = 193$$

To find d, subtract $a+b$ and $c+e$ from $a+b+c+d+e$. $a+b$ is the smallest of the pairwise sums, which is 67. $c+e$ is always the second largest of the pairwise sums, making it 85.

$$193 - 67 - 85 = \boxed{\textbf{(A) } 41}$$

Problem 15

Bocchi is playing a game where she traces a path through a given array of letters to form a word. If she can step vertically, horizontally, or diagonally to any adjacent letter, how many ways can she spell ONTARIO using this array?

(A) 1925 **(B)** 2035 **(C)** 2090 **(D)** 2145 **(E)** 2200

Solution: We begin by noticing that each character other than O can only be used once, which means that from any letter, Bocchi can never move to the left, so we will trace the path as follows:

O	O	O	O(55)	O(110)	O(110)	O(55)
O	N(5)	N(3)	N(3)	I(55)	I(55)	O(165)
O	N(3)	T(17)	T(6)	R(55)	I(110)	O(330)
O	N(3)	T(9)	A(55)	R(55)	I(165)	O(385)
O	N(3)	T(17)	T(6)	R(55)	I(110)	O(330)
O	N(5)	N(3)	N(3)	I(55)	I(55)	O(165)
O	O	O	O(55)	O(110)	O(110)	O(55)

The number in each box represents the number of ways to reach that square starting from any O. This can be calculated as the sum of the values of the squares adjacent to it which contains the previous letter. We will add the number of paths to trace O $\to$ N $\to$ T $\to$ A $\to$ R $\to$ I $\to$ O at each ending O to find the total number of paths.

$55 + 110 + 110 + 55 + 165 + 330 + 385 + 330 + 165 + 55 + 110 + 110 + 55 = \boxed{\textbf{(B) } 2035}$

Problem 16

If $3x^2 - qx + r = 0$, where q and r are prime numbers, has distinct rational roots, what is the product of all possible values of q?

(A) 14 **(B)** 15 **(C)** 21 **(D)** 33 **(E)** 35

Solution: Since r is prime, there are only 2 possible factorizations of the quadratic:

$$3x^2 - qx + r = (3x - r)(x - 1)$$
$$3x^2 - qx + r = (3x - 1)(x - r)$$

For the first case, we have $q = r + 3$, and since the only even prime is 2, the only solution is $r = 2, q = 5$. For the second case, we have $q = 3r + 1$, which again only has solutions when $r = 2$, since otherwise q would be even. This gives $r = 2, q = 7$. The product of all possible values of q is $5 \cdot 7 = \boxed{\textbf{(E)} 35}$.

Problem 17

If $\sin\theta + \cos\theta = \frac{\sqrt{2}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is the value of $\tan\theta - \cot\theta$?

(A) $-\frac{\sqrt{3}}{2}$ **(B)** $-\frac{8\sqrt{6}}{23}$ **(C)** $-\frac{\sqrt{6}}{3}$ **(D)** $\frac{2}{25}$ **(E)** $\frac{8\sqrt{6}}{23}$

Solution: Squaring the equation, we see that

$$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{2}{25} \tag{1}$$
$$2\sin\theta\cos\theta = \frac{2}{25} - 1 \tag{2}$$
$$\sin 2\theta = -\frac{23}{25} \tag{3}$$

Then we can solve for $\cos 2\theta = -\frac{4\sqrt{6}}{5}$. We can also simplify $\tan\theta - \cot\theta$:

$$\tan\theta - \cot\theta = \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} \tag{4}$$
$$= -\frac{2\cos 2\theta}{\sin 2\theta} \tag{5}$$

Substituting the values of $\sin 2\theta$ and $\cos 2\theta$ gives an answer of $\boxed{\textbf{(B)}\ -\frac{8\sqrt{6}}{23}}$.

Problem 18

A sequence of integers $a_1, a_2, \cdots, a_{2023}$ satisfies $a_n = 3a_{n-1} - 2a_{n-2}$ for all $n \geq 3$, with $a_1 = 7$ and $a_2 = 13$. Find the remainder of $a_1 + a_2 + \cdots + a_{2023}$ when divide by 5.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: Consider the sequence modulo 5. Computing the first few terms, we get

$$a_1 = 2, a_2 = 3, a_3 = 0, a_4 = 4, a_5 = 2, a_6 = 3\cdots$$

The sequence repeats every 4 terms, for a total of 505 repeats up to 2020. There are 3 terms left over, $a_{2021} = 2, a_{2022} = 3, a_{2023} = 0$. The total sum is then

$$505(2+3+0+4)+2+3+0 \equiv \boxed{\textbf{(A)}\ 0} \mod 5.$$

Problem 19

The function $f(x)$ is defined over the positive integers, and $f(1) = 0$. If x is divisible by 3, then $f(x) = f(\frac{x}{3}) + 3$. Otherwise, $f(x) = 3f(x-1)$. Find the value of $\frac{f(3^{2025}-1)}{f(3^{1013}-1)}$.

(A) $3^{2023}-1$ **(B)** $3^{2023}+1$ **(C)** $3^{2023}+4$ **(D)** $3^{2024}-1$ **(E)** $3^{2024}+1$

Solution: Let $g(n) = f(3^n - 1)$ for ease of notation. We can see that

$$g(n) = f(3^n - 1) = 9f(3^n - 3) = 9f(3^{n-1} - 1) + 27 = 9g(n-1) + 27$$

Additionally, $f(3-1) = 3f(3-2) = 0$. By recursion, we can find the general form of $g(n)$:

$$g(n) = 27(1 + 3^2 + 3^4 + \cdots + 3^{2n-4}) = 27\left(\frac{3^{2n-2} - 1}{8}\right)$$

Then we can evaluate the desired expression:

$$\begin{aligned}\frac{f(3^{2025} - 1)}{f(3^{1013} - 1)} &= \frac{g(2025)}{g(1013)} \\ &= \frac{3^{4048} - 1}{3^{2024} - 1} \\ &= \boxed{\textbf{(E) } 3^{2024} + 1}\end{aligned}$$

Problem 20

For any positive integer n, let $\omega(n)$ denote the number of non-negative integers m where $2^m < n$ such that $\lfloor \frac{n}{2^m} \rfloor$ is even. Find $\sum_{i=1}^{1023} \omega(i)$. ($\lfloor x \rfloor$ is the largest integer less than or equal to x)

(A) 2023 **(B)** 3072 **(C)** 4097 **(D)** 8193 **(E)** 10240

Solution: Notice that $\lfloor \frac{n}{2^m} \rfloor + 2 = \lfloor \frac{n+2^{m+1}}{2^m} \rfloor$ has the same parity as $\lfloor \frac{n}{2^m} \rfloor$. If we define a function $f(n)$ which returns 1 if $\lfloor \frac{n}{2^m} \rfloor$ is even and 0 of $\lfloor \frac{n}{2^m} \rfloor$ is odd for a fixed value of m, we see that $f(n)$ is in fact periodic with period 2^{m+1}! Additionally, we know each period is divided into exactly 2^m 1s and 2^m 0s. For example, for $m = 2$, the pattern goes $0000111100001111\cdots$

When n is one less than a power of 2, say $2^k - 1$, the value of $\omega(n)$ is exactly 0. What this implies is that for all values $m < k$, $f(n)$ where $n < 2^k$ has essentially finished a whole number of periods by $2^k - 1$, since the 0s do not affect the sum of $f(n)$. The restriction that $n > 2^m$ means that the number of periods completed can be calculated as $\lfloor \frac{2^k - 2^m}{2^{m+1}} \rfloor = 2^{k-m-1} - 1$ for each value of m. We multiply by 2^m, which is the sum of $f(n)$ over each period. Taking the sum over all possible values of m,

$$\sum_{i=0}^{k-1} 2^{k-1} - 2^m = (k-2)2^{k-1} + 1.$$

The question asks to sum up to 1023, which is $2^{10} - 1$, so $k = 10$. Plugging this into the formula, the sum is $\boxed{\textbf{(C) } 4097}$.

Problem 21

In a store, there is a collection of strange bills. There is a red and a green bill of value $\$2^i$, for all natural numbers i. There is only one \$1 bill. The *n-score* is defined to be the number of ways to form $\$n$ using the collection of strange bills. An *n-score* is said to be *tricky* if it is divisible by 2023. Find the number of positive integers between 1 and 5000 for which the *n-score* is tricky.

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Solution: Consider the polynomial

$$(1+x)(1+x^2)^2(1+x^4)^2(1+x^8)^2\cdots.$$

Then, notice that the $n-$score is the coefficient of x^n in the above polynomial. The polynomial is can be simplified as follows:

$$\begin{aligned}(1+x)(1+x^2)^2(1+x^4)^2(1+x^8)^2\cdots &= \frac{(1+x)^2(1+x^2)^2(1+x^4)^2(1+x^8)^2\cdots}{1+x}\\ &= \frac{(1+x+x^2+x^3+\cdots)^2\cdots}{1+x}\\ &= \frac{1}{(1-x)^2(1+x)}\\ &= \frac{1}{4}\left(\frac{1}{1+x}+\frac{1}{1-x}+\frac{1}{(1-x)^2}\right).\end{aligned}$$

Note that the second equality is because every number can be uniquely written as the sum of some powers of 2. The last equality uses the technique of partial fractions. The expression can be further simplified to

$$\frac{1}{4}\left(\frac{2}{1-x^2}+\frac{2}{(1-x)^2}\right).$$

Since $\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$, either by squaring the equation $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$, using the extended binomial theorem for $n \in \mathbb{Q}$, or using the power series for $\frac{1}{(1-x)^2}$, the expression is equal to

$$\frac{1}{4}\left((2+2x^2+2x^4+2x^6+\cdots)+(2+4x+6x^2+8x^3+\cdots)\right).$$

Simplifying, this becomes

$$\frac{1}{2}\left((1+x^2+x^4+x^6+\cdots)+(1+2x+3x^2+4x^3+\cdots)\right),$$

which upon combining terms becomes

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7\cdots.$$

Continuing this pattern, the coefficients are increasing for every two terms. This yields that $n=4044$ and $n=4045$ giving a coefficient of 2023. The values of n that yield a coefficient of 4046 are too large and are outside of the range. There are no values of n for which the coefficient of x^n is 0. Hence, the answer is $\boxed{\textbf{(B) } 2}$.

Problem 22

Let S be the sum of the number of elements in $A \cup B$ for all unordered pairs of distinct subsets A and B of $\{1, 2, 3, \cdots, 2023\}$. Find the remainder when S is divided by 5. ($A \cup B$ denotes the set of elements which are in either or both of A and B)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: Let $L = \{1, 2, 3, \cdots, 2023\}$. Note that the number of subsets of L is 2^{2023}, since every element can either be in the subset or not.

The number of subsets of L that does not contain 1 is 2^{2022}, by symmetry. The total number of pairs of subsets of L is $\binom{2^{2023}}{2}$, and the number of pairs of subsets that each do not contain 1 is $\binom{2^{2022}}{2}$. The number of pairs of subsets which contains 1 in the union is

$$\begin{aligned}\binom{2^{2023}}{2} - \binom{2^{2022}}{2} &= \frac{2^{2023}(2^{2023}-1) - 2^{2022}(2^{2022}-1)}{2} \\ &= 2^{2021}(3 \times 2^{4043} - 1).\end{aligned}$$

Since the element 1 appears in the union of this many pairs of subsets, it contributes this value towards S as well. By symmetry, each other element contributes this much as well. The value of S is then

$$2023 \times 2^{2021}(3 \times 2^{2022} - 1)$$

Using Fermat's Little Theorem, the remainder is $\boxed{\textbf{(B) } 1}$ mod 5

Problem 23

A binary string is said to be *reflectable* if it is a palindrome after all leading zeros and trailing zeros have been removed. For example, 011011000, 000, and 0100 are all *reflectable*. If the number of *reflectable* 2023-digit binary strings can be represented as $5(2^b) - c$, where b is minimized, and a, b are both positive integers, find $a + b$.

(A) 1013 **(B)** 4178 **(C)** 5069 **(D)** 6072 **(E)** 6076

Solution: Consider all palindromes of length n. $2023 - n$ 0s are added to either side to make it a 2023 digit reflectable binary string. Note that the first and last digits must be 1, or else it would effectively be a palindrome of length $n - 2$. There are $2^{\lfloor \frac{n-1}{2} \rfloor}$ palindromes of size n, and there are $2024 - n$ ways to add zeros around it. Note the additional case where all digits are 0. Taking the sum over all possible lengths and using the formula for the sum of a geometric sequence,

$$\begin{aligned}\sum_{i=1}^{2023}(2024-n)2^{\lfloor\frac{n-1}{2}\rfloor}+1 &= \left(\sum_{i=0}^{1011}(4045-4i)2^{i}\right)+1\\ &= 4\left(\sum_{i=1}^{1011}2^{i}-1\right)+2^{2012}\\ &= 4\left(2^{1011}+2^{1010}+\cdots+2+1-1012\right)+2^{2012}\\ &= 4\left(2^{1012}-1013\right)+2^{2012}\\ &= 5\times2^{1012}-4052\end{aligned}$$

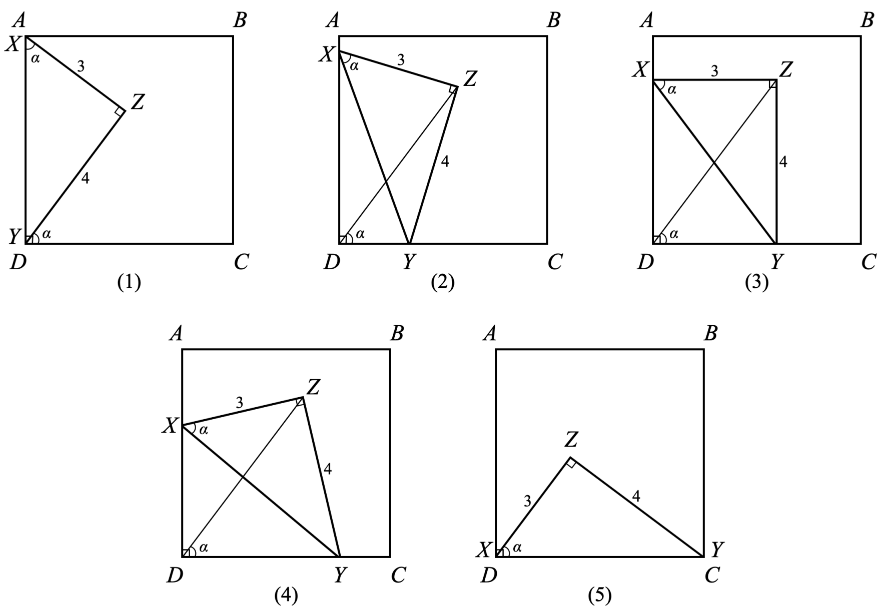
The answer is $1012+4052=$ $\boxed{\textbf{(B) } 5064}$

Problem 24

In the diagram, $ABCD$ is a square box, and XYZ is a triangle with right angle at Z and side lengths 3, 4, and 5. Initially, X and Y lie on A and B, respectively, as shown. The triangle slides counter-clockwise inside the box such that X and Y always remain on the side of $ABCD$. What is the total distance traveled by Z if XYZ moves once around the square back to its original position?

(A) $2\sqrt{2}\pi$ **(B)** $6\sqrt{3}$ **(C)** 3 **(D)** 12 **(E)** $2\sqrt{5}\pi$

Solution:



In diagrams (2), (3), and (4), quadrilateral $XZYD$ is cyclic, since $\angle XDY+\angle XZY=90^\circ+90^\circ=180^\circ$. Since both angles are subtended by the same arc ZY, $\angle ZXY=\angle ZDY=\alpha$, and so we observe that the point Z always travels on a straight line that makes a constant angle α with DC.

We can divide the path of Z in a quarter cycle into two distinct phases: upward movement and downward movement.

The greatest ZD occurs when XZ is horizontal and parallel to DC, as shown in (3). Here, the quadrilateral $XZYD$ forms a rectangle, with XY and ZD as the diagonals. Hence $XY = ZD = 5$, since the diagonals of a rectangle are equal in length.

Since Z always travels on a straight line, from (1) to (3), the distance travelled upwards by Z can be calculated by the difference of ZD at (1) and ZD at (3). Since ZD is coincident with ZY at (1), $ZD_{(1)} = ZY_{(1)} = 4$. Hence the distance travelled by Z from (1) to (3) is $a = |ZD_{(3)} - ZD_{(1)}| = |5 - 4| = 1$.

We can use the same method to calculate the distance travelled by Z downwards from (3) to (5). Since ZD is coincident with ZX at (5), $ZD_{(5)} = ZX_{(5)} = 3$. Hence the distance travelled by Z from (3) to (5) is $b = |ZD_{(5)} - ZD_{(3)}| = |3 - 5| = 2$.

The combined distance traveled by Z in a quarter cycle is $a + b = 3$. Therefore the total distance traveled by Z in a full cycle is $3 \cdot 4 = \boxed{\textbf{(D)}\ 12}$.

Problem 25

Find the sum of the digits of the square of a number formed by 2023 "1"s, i.e. $111 \cdots 111^2$.

(A) 18180 **(B)** 18193 **(C)** 18208 **(D)** 18225 **(E)** 18226

Solution: First, we begin by replacing the square with a term that allows the expression to be expanded into two terms we can analyze separately.

$$\begin{aligned}\underbrace{11\cdots11^2}_{2023\,digits} &= \underbrace{11\cdots11}_{2023\,digits}\left(\frac{1}{9}(10^{2023}-1)\right)\\ &= \frac{1}{9}\times\underbrace{11\cdots11}_{2023\,digits}\times10^{2023} - \frac{1}{9}\times\underbrace{11\cdots11}_{2023\,digits}\end{aligned}$$

Setting $x = \underbrace{123456790\cdots123456790}_{224\,times}123456$, the expression becomes

$$\left(x \times 10^{2023} + \frac{7}{9}\times 10^{2023}\right) - \left(x + \frac{7}{9}\right)$$

The sum of the digits of $x \times 10^{2023}$ can be evaluated as $224 \times 37 + 21 = 8309$. Now we need to find the sum of the digits of $\frac{7}{9} \times 10^{2023} - x + \frac{7}{9}$:

$$\begin{array}{r} 7777777777\cdots7777777777777777.77\cdots \\ -\underline{0123456790\cdots123456790123456.77\cdots} \\ 7\underbrace{654320987\cdots654320987}_{224\,times}654321.00\cdots \end{array}$$

Notice that this is only 2023 digits, so it does interfere with the digits of $x \times 10^{2023}$ from earlier. Evaluating the sum of digits, $224 \times 44 + 21 + 7 = 9884$. Thus, the total sum of digits of $\underbrace{11\cdots11^2}_{2023\,digits}$ is $8309 + 9884 = \boxed{\textbf{(B)}\ 18193}$.