OMC WOLFRAM AoPS TTmath Top Math School (www.ttmath.ca)

Ontario Mathematics Competition 2023

(Part II: Long Answer)

Monday, March 20, 2023 -
Friday, March 31, 2023

Solutions

Problem 1

(a) Find the value of x for which $\frac{5}{3} + \frac{13}{x} = 2$.

(b) Find all real values of x for which $\frac{5}{x} + \frac{7}{x+1} = \frac{19}{x(x+1)}$ where $x \neq 0$ and $x \neq -1$.

(c) Find all real values of x for which $\frac{3}{x^2} + \frac{5}{x} = 2$.

Solution (a) (Total marks: 8):

$2 - \frac{5}{3} = \frac{1}{3}$

$x = 3 \times 13$

$x = 39$

Solution (b) (Total marks: 8):

$5(x+1) + 7x = 19$

$12x + 5 = 19$

$x = \frac{7}{6}$

Solution (c) (Total marks: 14):

$3 + 5x = 2x^2$

$2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$ or $x = \frac{5 \pm \sqrt{49}}{4}$ or $x = \frac{5 \pm 7}{4}$.

$x = -\frac{1}{2}, 3$

Problem 2

Define the *n-adjacent* set, A_n, as the set of the first n positive integers, i.e. $A_n = 1, 2, 3, \cdots, n-1, n$.
(a) How many 9 digit integers using each element of A_9 exactly once in its digits are divisible by 4?
(b) If set B and C each consist of 8 random elements from the set A_9, what is the probability that the difference between the sums of all elements of B and C is divisible by 4?
(c) Find the smallest positive integer value of n such that the average value of $|x-y|$ for all order pairs (x, y) for distinct x and y in A_n is at least 2023.

Solution to (a) (Total marks: 7)
A number is divisible by 4 if its last two digits are divisible by 4. Excluding numbers that use 0 or repeated digits, the number of two digit integers which are divisible by 4 is 16. The other digits can be arranged in any order, so the answer is

$$16 \times 7! = 80640.$$

Solution to (b) (Total marks: 8)
Instead of choosing 8 random elements, think of it as removing 1 random element. For the difference between the sums of B and C to be divisible by 4, the element removed from each of the two sets must have the same remainder when divided by 4. The number of ways to choose such elements is

$$3^2 + 3(2^2) = 21.$$

The probability is then $\frac{21}{81} = \frac{7}{27}$.

Solution to (c) (Total marks: 15)
To calculate the average value, we can consider each possible difference, and how many times it occurs. For each difference x, the number of pairs which gives this difference is $n - x$. This yields the sum

$$1(n-1) + 2(n-2) + \cdots + (n-1)1 = \frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}$$

Dividing this by the number of pairs, which is $\binom{n}{2} = \frac{n(n-1)}{2}$, gives the average. This gives

$$\frac{\frac{n(n-1)}{2}\left(\frac{3n-(2n-1)}{3}\right)}{\frac{n(n-1)}{2}} = \frac{n+1}{3}$$

This average needs to be at least 2023, so we write

$$\frac{n+1}{3} \geq 2023 \implies n \geq 6068.$$

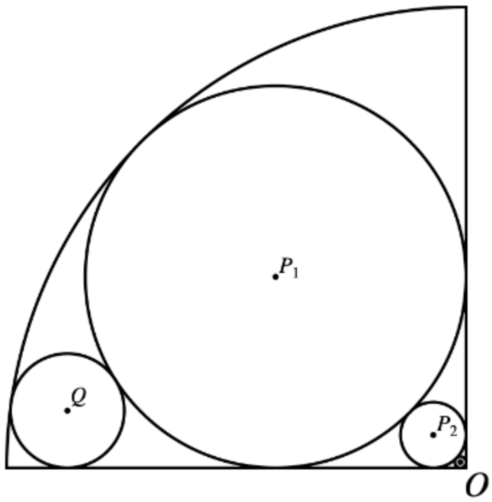
Problem 3

For each of the circles in the infinite set P, P_n is externally tangent to P_{n-1} and is tangent to the radii of the quarter circle O, as shown in the diagram. P_1 has radius 4.

(a) Find the radius of the quarter circle O.

(b) Find the sum of the areas of all circles in the set P.

(c) Draw a circle Q so that it is externally tangent to P_1, internally tangent to the arc of O, and tangent to a radius of O, as shown in the diagram. Find the radius of Q. (express your answer as $a\sqrt{b}+c$ for rational a and c and integer b)



Solution to (a) (Total marks: 5)

Consider the line segment from the center of P_1 to the center of O. The length of this segment is $4\sqrt{2}$ since it is the hypotenuse of an isosceles right triangle with leg length 4. Then the radius of O is $4+4\sqrt{2}$.

Solution to (b) (Total marks: 9)

Consider any two circles P_n and P_{n+1} in P. We want to find the relationship between their areas, to find a general trend for what the sum of all areas converges to. Let their radii be r_n and r_{n+1} respectively. As shown in the solution of part a, the distance from the center of P_n is $r_n\sqrt{2}$. Doing the same for P_{n+1}, we can describe the distance $\overline{P_nO}$ in two ways and equate them:

$$r_n\sqrt{2} = r_n + r_{n+1} + r_{n+1}\sqrt{2} \implies r_{n+1} = r_n\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

Square both sides and rationalize to get the relation between the areas:

$$A_{n+1} = A_n(17-12\sqrt{2})$$

Thus the areas of the circles forms a geometric sequence! The area of the first circle, P_1, is 16π. We can deal with this using the sum of an infinite geometric sequence:

$$\begin{aligned}16\pi(1+(17-12\sqrt{2})+(17-12\sqrt{2})^2+\cdots) &= \frac{16\pi}{1-(17-12\sqrt{2})}\\ &= 8\pi+6\sqrt{2}\pi\end{aligned}$$

Solution to (c) (Total marks: 16)
Let the centers of Q and P_1 be O_Q and O_P, respectively. Let the point of tangency between Q and the radius of O be A, and the point of tangency between P_1 and the radius of O closest to Q be B. The foot of the perpendicular from O_Q onto $\overline{O_P B}$ is C. Let the radius of Q be r.
By the pythagorean theorem,

$$\overline{AB} = \overline{O_Q C} = (r+4)^2 - (4-r)^2 = 4\sqrt{r}$$

Since $\overline{O_Q O} + r$ is equal to the radius of O, we can write

$$\begin{aligned} \sqrt{r^2 + (4 + 4\sqrt{r})^2} + r &= 4 + 4\sqrt{2} \\ r^2 + 16 + 32\sqrt{r} + 16r &= 48 + r^2 + 32\sqrt{2} - 8r - 8\sqrt{2}r \\ (3+\sqrt{2})r + 4\sqrt{r} - 4(1+\sqrt{2}) &= 0 \end{aligned}$$

This is now a quadratic in $\sqrt{r}$, and solving gives $\sqrt{r} = \sqrt{2} - 1$, or $r = 3 - 2\sqrt{2}$. The other solution to the quadratic yields a negative value of r, which is impossible.

Problem 4

(a) A *joli function* f is a function defined over the real numbers that satisfies $f(0) = -1$ and $(f(x) - f(y) - \sin x + \sin y)(f(x) - f(y) - x^2 + y^2) = 0$ for all real x and y. Find all *joli functions*.
(b) A *verminous function* g is an even function defined over the real numbers that satisfies $g(g(x)) + 2yg(x) + g(y) = g(x^2 + y)$ for all real x and y. Find all *verminous functions*.
(c) Find all ordered pairs of integers (x, y) such that $|x|, |y| \leq 360$ that satisfies

$$g(f(x))(g(y) + 3)(g(y) - 3) + 4f(x)(g(y)^2 + 108) + 4g(g(y)) = 5184$$

for some *joli function* f and *verminous function* g. (All trigonometric functions are in degrees)

Solution to (a) (Total marks: 6)
Solving the first bracket gives $f(x) = \sin x + c$ for a constant c, and the second bracket gives $f(x) = x^2 + c$. Using the condition that $f(0) = -1$, the two choices for $f(x)$ becomes $f(x) = \sin x - 1, x^2 - 1$. Now we show that a piecewise function does not work. Assume that $f(a) = \sin(a) - 1$ for some value of a and $f(b) = b^2 - 1$ for a value of $b \neq a$. Additionally, $\sin(a) \neq a^2, \sin(b) \neq b^2$ since otherwise both functions give the same value at that point, so the piecewise condition would not matter. Then setting $x = a, y = b$,

$$(\sin(a) - a^2)(\sin(b) - b^2) = 0$$

Which is a contradiction.

Solution to (b) (Total marks: 14)
When solving a functional equation, the idea is often to plug in values of x and y to extract as much information as possible. The first step is often determining the value of $g(0)$, so we try $x = y = 0$:

$$g(g(0)) = 0$$

This is not what we want, but this incentivizes the substitution of $x = 0, y = g(0)$:

$$2g(0)^2 = 0 \Longrightarrow g(0) = 0$$

Now that we got the result of $g(0) = 0$, we want to use this to cancel out some terms by substituting $y = 0$:

$$g(g(x)) = g(x^2)$$

Now we can try $y = -x^2$ to make the RHS 0. Using the fact that $g(x^2) = g(-x^2)$ since g is even to combine terms on the LHS, we get

$$g(x^2) = x^2 g(x)$$

From $g(xy) = g(x)g(y)$ we can derive $g(x^2) = g(x)^2$, and substitute this into the previous equation:

$$g(x)^2 = x^2 g(x)$$

Finally, either $g(x) = 0$ or $g(x) = x^2$.

Solution to (c) (Total marks: 15)
First, if $g(y) = 0$, the equation simplifies to

$$f(x) = 12$$

which has no solutions for either of the possible functions f. Otherwise, $g(x) = x^2$. In this case, we can group the terms and factor:

$$\begin{aligned}
f(x)^2(y^4 - 9) + 4f(x)(y^4 + 108) + 4y^4 - 5184 &= 0 \\
(y^4 f(x)^2 + 4y^4 f(x) + 4y^4) - (9f(x)^2 - 432f(x) + 5184) &= 0 \\
(y^2 f(x) + 2y^2)^2 - (3f(x) - 72)^2 &= 0 \\
(y^2 f(x) + 2y^2 + 3f(x) - 72)(y^2 f(x) + 2y^2 - 3f(x) + 72) &= 0 \\
((f(x) + 2)(y^2 + 3) - 78)((f(x) + 2)(y^2 - 3) + 78) &= 0
\end{aligned}$$

Case 1: $(f(x) + 2)(y^2 + 3) = 78$
If $f(x) = \sin(x) - 1$, then we have

$$(\sin x + 1)(y^2 + 3) = 78$$

By Niven's Theorem, if $sin(x)$ is rational, then x is also rational if and only if $|\sin x| = 0, \frac{1}{2}, 1$. This means the possible rational values of $(\sin x + 1)$ are

$$\sin x + 1 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

If each of these cases are checked, the following solutions for a pair $(\sin x, y^2)$ are found: $(\frac{1}{2}, 49), (1, 36)$. The solution pairs (x, y) are then

$$(30, \pm 7), (150, \pm 7), (-210, \pm 7), (-330, \pm 7), (90, \pm 6), (-270, \pm 6)$$

If $f(x) = x^2 - 1$, then the equation becomes

$$(x^2 + 1)(y^2 + 3) = 78$$

and testing values of y from 0 to 8 gives the solutions

$$(\pm 5, 0), (\pm 1, \pm 6).$$

Case 2: $(f(x) + 2)(y^2 - 3) = -78$
If $f(x) = \sin x - 1$, there are no solutions because then $y^2 - 3$ has to be negative, and it is not large enough for the equation to be true.
If $f(x) = x^2 - 1$, checking solutions yields $(x, y) = (\pm 5, 0)$, which is a duplicate of a previous case.
The list of all possible solutions for (x, y) are

$$(30, \pm 7), (150, \pm 7), (-210, \pm 7), (-330, \pm 7), (90, \pm 6), (-270, \pm 6), (\pm 5, 0), (\pm 1, \pm 6).$$