OCMC

Solutions

Ontario Invitational Mathematics Exam

2024

1. Let α be the number that is created by concatenating the first 26 positive integers, $\alpha = 1234567\ldots242526$. What is the remainder when α is divided by 99?

   proposed by: Shanna Xiao

   Solution: Note that $100 \equiv 1 \pmod{99}$, so we have

$$\begin{aligned}\alpha &\equiv 1 + 23 + 45 + 67 + 89 + \sum_{i=10}^{26} i &\pmod{99}\\ &\equiv 27 + \frac{26 \times 27}{2} - \frac{9 \times 10}{2} &\pmod{99}\\ &= \boxed{36} &\pmod{99}\end{aligned}$$

2. 10 students, labeled P_0, P_1, to P_9 are on their way to a math contest when one of them robs a bank. They are now being interrogated and make up two statements. A student is a liar if and only if at least one of their two statements is a lie. Every student P_i says, "P_{i+1} is a liar" and "P_{i+2} or P_{i+3} robbed the bank" (taking indices modulo 10, so $P_{10} = P_0$). How many of the 20 statements are true?

   proposed by: Christopher Li

   Solution: Without loss of generality, assume P_3 is the robber. We can do this because we can relabel the students cyclically. Then all of P_2 to P_9 must be liars because their second statements is false. P_1 is not a liar because both of their statements are true. P_0 is a liar because their first statement is a lie (P_1 is not a liar). Note that since P_0 and $P_2, \dots P_9$ are all liars, the first statement of $P_2, \dots P_9$ are all true. Thus everyone has said one true statement except P_1 who has two true statements. The total number of true statements is $2 + 9 = \boxed{11}$.

3. Shanna and Elaine are in the airport, and they are standing at the left end of a 108m moving walkway, which is moving right at a speed of 2m/s. Shanna walks beside the moving walkway on the regular ground (the ground travels at 0m/s) at constant speed. Elaine is very energetic. She runs on the moving walkway at 4m/s to the right end of walkway, turns back, and runs on the moving walkway until she meets Shanna. Then she immediately turns back and repeats this. However, she gets tired after doing this twice, so she runs for $\frac{50}{9}$ seconds to the right end of the walkway and waits for Shanna. Find how fast Shanna walks, in m/s.

   proposed by: Elaine Li

   Solution: Let Shanna walk at xm/s. Let k be the current distance between Shanna and the right end of the walkway.

   When Elaine is running towards the right end of the walkway, she is running at $4 + 2 = 6$m/s compared to ground. It would take her $\frac{k}{6}$ seconds to get there. Shanna walked $\frac{kx}{6}$ meters towards the right end of the walkway, so the new distance between Shanna and the walkway is $k - \frac{kx}{6} = \frac{(6-x)k}{6}$.

   When Elaine is running from the right end of the walkway to Shanna, she is running opposite to the direction the walkway is moving in. Thus, she is running at $4 - 2 = 2$m/s. It would take $\frac{k}{2+x}$ seconds for them to meet, and Shanna walks $\frac{kx}{2+x}$ meters. The new distance between Shanna and the walkway is $k - \frac{kx}{2+x} = \frac{2k}{2+x}$.

   Thus, after Elaine runs towards the right twice and runs from the right to meet Shanna twice, the distance between Shanna and the walkway is

$$\left(\frac{6-x}{6}\right)^2 \times \left(\frac{2}{2+x}\right)^2 \times 108 = 12 \times \frac{(6-x)^2}{(2+x)^2}$$

   After this, since Elaine runs for $\frac{50}{9}$ seconds in the end to get to the right end, she ran $\frac{50}{9} \times 6 = \frac{100}{3}$ meters. Thus, Shanna was $\frac{100}{3}$ meters away from the right end, so

$$\begin{aligned} 12 \times \frac{(6-x)^2}{(2+x)^2} &= \frac{100}{3} \\ \frac{(6-x)^2}{(2+x)^2} &= \frac{25}{9} \\ 9(x^2 - 12x + 36) &= 25(x^2 + 4x + 4) \\ x^2 + 13x - 14 &= 0 \\ (x+14)(x-1) &= 0 \end{aligned}$$

   Since Shanna's speed is not negative, she must be walking at x = $\boxed{1m/s}$.

4. Let ellipse C be defined by the equation

$$\frac{x^2}{25} + \frac{y^2}{27} = 1.$$

Suppose $\triangle ABC$ is a triangle that completely covers the ellipse. Determine the minimum value of the area of $\triangle ABC$.

proposed by: Terry Yang

Solution: Applying affine transformation,

$$\begin{cases} x' &= x \\ y' &= \frac{5}{3\sqrt{3}} y \end{cases}$$

Therefore, ellipse C is transformed into circle $O : x^2 + y^2 = 25$.

Let S be the smallest area of the original triangle ABC, and S' be the smallest area of the transformed triangle $A'B'C'$

Thus, $S' = \frac{5}{3\sqrt{3}} S$. The smallest triangle covering a circle is a equilateral triangle, which gives,

$$S'_{min} = \frac{\sqrt{3}}{4}(2\sqrt{3} \times 5)^2 = 75\sqrt{3}$$

Therefore,

$$S_{min} = \frac{3\sqrt{3}}{5}\left(75\sqrt{3}\right) = \boxed{135}.$$

For completeness, let's prove that the smallest triangle covering a circle is equilateral. Let s be the semi-perimeter and a, b, c be the side lengths of the triangle. By the AM-GM inequality, we have

$$(s-a)(s-b)(s-c) \leq \left(\frac{(s-a)+(s-b)+(s-c)}{3}\right)^3 = \frac{s^3}{27}.$$

Writing the area of $\triangle ABC$ through both Heron's formula and the inradius formula, we have $5s = \sqrt{s(s-a)(s-b)(s-c)}$. This gives

$$\frac{s^4}{27} \geq 25s^2 \Longleftrightarrow s \geq 15\sqrt{3}.$$

The area of the triangle in this case is $5s = 75\sqrt{3}$ which is the same as when the triangle is equilateral.

5. Square S has vertices at $(0,0),(6,0),(0,6)$, and $(6,6)$. If two lattice points are chosen at random in the interior of S, what is the probability that the line passing through the two points divides S into two regions with equal area?

   proposed by: Daniel Chen

   Solution: We will show that this line must pass through the center of the square at $(3,3)$. If the line crosses two adjacent sides of the square, then the triangle formed by the line and the two sides it crosses has area less than half of the square, which is not possible. If the square instead crosses two opposite sides of the square, then without loss of generality assume that the two sides it crossed are $x=0$ and $x=6$. Let the points of intersection be $(0,a)$ and $(6,b)$. In order for the area of the quadrilateral formed by $(0,0),(6,0),(6,b),(0,a)$ to be half of the square, we must have $6\times\frac{a+b}{2}=18$. But then the midpoint of $(0,a)$ and $(6,b)$ is $(3,\frac{a+b}{2})=(3,3)$ which is the center of the square.

   For the other direction, we can check that as long as the line passes through the center, it will split the square into two regions of equal area. This is clear by symmetry.

   Now we count the number of pairs of points that work. Clearly the center pairs with any other point, which gives 24 options. There are 4 points other than the center on each of the 4 axis of symmetry of the square, which gives $4\times\binom{4}{2}=24$ options. Finally, all the remaining points can pair with their reflection across $(3,3)$ which yields 4 options.

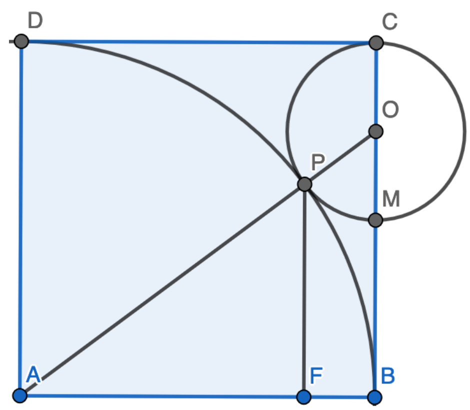
   Since there are a total of $\binom{25}{2}=300$ pairs of points, the probability is

$$\frac{24+24+4}{300}=\boxed{\frac{13}{75}}.$$

6. Let E be a point in unit square $ABCD$ such that $AE = 1$. Let M be the midpoint of BC. What is the distance of E to AD when $\angle CEM$ is maximized?

   proposed by: Shanna Xiao

   Solution:



Let ω_1 be the circle with radius 1 centered at A, and ω_2 be the circle with diameter MC. Let r_1 and r_2 be the radius of ω_1 and ω_2 respectively, and define O as the center of ω_2. Clearly E lies on ω_1. We will show that ω_1 and ω_2 are externally tangent. Indeed, by the Pythagorean Theorem we can check that

$$(r_1 + r_2)^2 = \left(1 + \frac{1}{4}\right)^2 = 1^2 + \left(\frac{3}{4}\right)^2 = AB^2 + BO^2.$$

Let this tangency point be P. We can check that $\angle CEM$ is maximized exactly when $E = P$ since if E is outside of ω_2, then $\angle CEM$ will be acute.

It remains to find the distance from P to AD. Let the foot of the altitude from P to AB be F. Then $\triangle APF \sim \triangle AOB$, so

$$AF = \frac{AP \times AB}{AO} = \boxed{\frac{4}{5}}.$$

7. Let $a_1, a_2, \dots a_{119}$ be a strictly increasing arithmetic sequence of positive integers with common difference d and $a_1 = 119$. Find the number of possible values $d \leq 119$ so that there exist positive integers $1 < i < j \leq 119$ such that $119, a_i, a_j$ forms a geometric sequence.

   proposed by: Daniel Chen

   Solution: Write $a_i = 119 + md$ and $a_j = 119 + nd$ with $0 < m < n \leq 118$. Since $119, a_i, a_j$ forms a geometric sequence, we can write

$$\begin{aligned} 119(119 + nd) &= (119 + md)^2 \\ m^2 d^2 &= 119d(n - 2m) \\ d &= \frac{119(n - 2m)}{m^2} \end{aligned}$$

   In order for $d \leq 119$ to hold, either $n - 2m = m^2$ or $\gcd(m, 119) \neq 1$. In the first case, simply taking $m = 1, n = 3$ works, which gives $d = 119$. In the second case, we must have $\gcd(m, 119) \in \{7, 17\}$ since $m < 119 = 7 \times 17$.

   If $7 \mid m$, we can assume WLOG that $m = 7$. This is because if $m = 7k$ for $k > 1$ and we have a solution for n, then k^2 must divide $n - 2m = n - 14k$. But then taking $m = 7$ and n_1 to be $\frac{n-14k}{k^2} + 14$ gives the same solution. Plugging this in, we get $d = \frac{17(n-14)}{7}$. Then it must be the case that $17 \mid d$, so there are 6 possible values of $d < 119$ which are the multiples of 17. We can check that for each of these values, a valid n exists.

   If $17 \mid m$, again assume WLOG that $m = 17$. We have $d = \frac{7(n-34)}{17}$ so $17 \mid (n - 34)$. Since $n \leq 119$, $\frac{n-34}{119} \leq 4$. There are then 4 possible values d in this case, namely $d = 7, 14, 21, 28$.

   In total, there are $1 + 6 + 4 = \boxed{11}$ possible values of d which are less than or equal to 119.

8. $P(x)$ is a polynomial with real coefficients such that some of its coefficients are negative and some are positive. If $P(x)^2$ has only positive or zero coefficients, what is the minimum possible degree of $P(x)$?

   proposed by: Pavel Mackenzie

   Solution: The answer is 4. $x^4 + 2x^3 - x^2 + 2x + 1$ is one such polynomial (squares to $x^8 + 4x^7 + 2x^6 + 11x^4 + 2x^2 + 4x + 1$). We will prove that any polynomial with lesser degree does not work.

   If the degree of $P(x)$ is at most 3, then it can be expressed as $ax^3 + bx^2 + cx + d$ (Note that the coefficients can be 0). We can expand this to

$$P(x)^2 = a^2x^6 + 2abx^5 + (b^2 + 2ac)x^4 + (2bc + 2ad)x^3 + (c^2 + 2bd)x^2 + 2cdx + d^2.$$

   If a and b or c and d have different signs, then either the x^5 or x term of $P(x)^2$ would have a negative coefficient. Thus a and b have the same sign and so do c and d. Then in order for $P(x)$ to have both positive and negative coefficients, it must be the case that a, b and c, d have opposite signs. But then the x^3 coefficient of $P(x)^2$ is negative, a contradiction.

   The only remaining case is when $2bc + 2ad = 0$. But bc and ad have the same sign, so $bc = ad = 0$. Suppose WLOG that $a = c = 0$. Then it must be the case that b and d are non-zero, so the coefficient of x^2 is negative, a contradiction.

9. K is an infinite sequence of positive integers such that the sum of any number of consecutive terms in K does not equal to 2025. Determine, with proof, the general expression for the minimum sum of the first n terms in sequence K.

   proposed by: Oscar Zhou

   Solution: We claim that the sequence with 2025-term cycles of $(1, 1, 1, \ldots, 1, 1, 2026)$ has the minimum sum for the first n terms. This sequence clearly satisfies the condition that no partial sum is equal to 2025.

   The sum of any 2025 consecutive terms in the proposed sequence is $2026 + 2024 \times 1 = 4050$. Therefore, it suffices to prove that the sum of any 2025 consecutive terms in any sequence K is greater than or equal to 4050.

   Suppose for the sake of contradiction that some sequence K has a sum of 2025 consecutive terms less than 4050. Let this 2025-term subsequence be $(a_1, a_2, \ldots, a_{2025})$ and define the sequence

$$(s_1, s_2, s_3, \ldots, s_{2025}) \text{ where } s_i = a_1 + a_2 + a_3 + \ldots + a_i.$$

   Thus, $s_i < 4050$ and $s_i \neq 2025$ for all i. Then, s_i mod 2025 can be any natural number between 1 and 2024, inclusive, since 2025 does not divide s_i. Since there are 2025 terms, by the pigeonhole principle, there exists s_j and s_k such that s_j is congruent to s_k mod 2025. WLOG, let $k > j$, then $2025 \mid (s_k - s_j)$, meaning that the sum of consecutive terms $a_{j+1}, a_{j+2}, \ldots, a_k$ is 2025, contradicting the original condition.

   Therefore, the minimum sum of the first n terms in sequence K is $\lfloor \frac{n}{2025} \rfloor$ cycles of terms that sum up to 4050 and $n - 2025 \times \lfloor \frac{n}{2025} \rfloor$ terms of 1 at the end. The general expression is $n + 2025 \times \lfloor \frac{n}{2025} \rfloor$.

10. Prove that there are infinitely many triples of positive integers (a, b, c) that satisfy the equation $a^2 + b^2 = 2c^2 + 1$.

    proposed by: Daniel Chen

    Solution:

    We will show that there are infinitely many solutions to the equation $a^2 + b^2 = d^2$ which simultaneously satisfy $d^2 = 1 + 2c^2$ using induction.

    The base case is $(a, b, c, d) = (8, 15, 12, 17)$ which works. Suppose that (a_i, b_i, c_i, d_i) is a valid solution. Note that $d^2 - 2c^2 = 1$ is Pell's equation with solutions generated by $(3 + 2\sqrt{2})^k$ for $k \in \mathbb{N}$, so $(c_{i+1}, d_{i+1}) = (2c_i d_i, c_i^2 + d_i^2)$ is another valid solution. Additionally, set $a_{i+1} = 4e_i^2 - 1$ and $b_{i+1} = 4e_i$. Then

$$\begin{aligned} a_{i+1}^2 + b_{i+1}^2 &= (4e_i^2 - 1)^2 + 16e_i^2 \\ &= 16e_i^4 + 8e_i^2 + 1 \\ &= (4e_i^2 + 1)^2 \\ &= (d_i^2 + 2e_i^2) \\ &= d_{i+1}^2. \end{aligned}$$

    so $(a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1})$ is also a solution, and it's easy to check that this is larger than (a_i, b_i, c_i, d_i) meaning we have a unique solution. We are done by induction.

    Remark: The motivation behind choosing these values of a_{i+1} and b_{i+1} comes from the characterization of Pythagorean triples: all Pythagorean triples can be written in the form $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$ for integers m, n. Since $d_{i+1} = 4e_i^2 + 1 = (2e_i)^2 + 1^2$, this choice is natural.