OCMC

Solutions

Ontario Mathematics Competition

2024

1. Compute the value of $\dfrac{2^{-1}}{(-1)^2} + \dfrac{3^2}{2^3}$.

   (A) 2 **(B)** $\dfrac{3}{2}$ **(C)** $-\dfrac{1}{9}$ **(D)** $-\dfrac{7}{8}$ **(E)** $\dfrac{13}{8}$

   proposed by: Pavel MacKenzie

   Solution:

$$\begin{aligned}\frac{2^{-1}}{(-1)^2} + \frac{3^2}{2^3} &= \frac{1}{2} + \frac{9}{8} \\ &= \boxed{\textbf{(E) } \frac{13}{8}}.\end{aligned}$$

2. If the area of square S_1 is 16 and the area of square S_2 is 25, what is the ratio between the perimeter of S_1 and the perimeter of S_2?

   (A) $\dfrac{4}{5}$ **(B)** $\dfrac{16}{25}$ **(C)** 1 **(D)** $\dfrac{25}{16}$ **(E)** $\dfrac{5}{4}$

   proposed by: Daniel Chen

   Solution: Since S_1 has area 16, we know that the side length is 4. Similarly, the side length of S_2 is 5. The perimeter of a square is 4 times its side length, so we get

$$\frac{4 \times 4}{4 \times 5} = \boxed{\textbf{(A) } \frac{4}{5}}.$$

3. David solves five problems each Saturday and Sunday, and he solves six problems each weekday. During some number of consecutive days, David solved 70 problems. Which day of the week did he start solving the problems?

   (A) Monday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Sunday

   proposed by: Terry Yang

   Solution: Let the number of days that David solves 5 problems be x and the number of days when he solves 6 problems be y. Since each day David can solve either 5 or 6 problems, we get that $5x + 6y = 70$. Since 6 only divides $(70 - 2 \cdot 5)$ and $(70 - 8 \cdot 5)$, x must be either 2 or 8.

   But if during some consecutive days the number of Saturdays and Sundays is 8, then the number of weekdays is 5, which is impossible.

Therefore, the number of Saturdays and Sundays is 2, and the number of weekdays is 10. The Saturday and the Sunday must appear between weekdays. Thus, he started to solve problems on $\boxed{\textbf{(A)}\ Monday}$.

4. Let x and y be real numbers satisfying $\dfrac{4}{x} = \dfrac{y}{3} = \dfrac{x}{y}$. What is the value of x^3?

   (A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 54

   proposed by: Christopher Li

   Solution: Rearranging, we have $4y = x^2$ and $3x = y^2$. Multiplying the two equations, we get $xy = 12$, and $x^2 = 4y = 48/x$ gives $x^3 = \boxed{\textbf{(D)}\ 48}$.

5. At a business conference, there are 48 Canadians and 102 Americans. Every Canadian shook the hands of exactly three Americans, and each American either shook the hands of exactly two Canadians or did not shake hands with anyone. How many Americans did not shake hands with anyone?

   (A) 5 **(B)** 10 **(C)** 20 **(D)** 30 **(E)** 80

   proposed by: Oscar Zhou

   Solution: Let the number of Americans who did not shake hands be n. Since every handshake must count for exactly 1 Canadian and 1 American, we get the equation $3(48) = 2(102 - n)$. Solving the equation yields $n = \boxed{\textbf{(D)}\ 30}$.

6. Call a positive integer *prime-increasing* if its digits from left to right are strictly increasing and every pair of adjacent digits form a prime number. What is the sum of the digits of the largest prime-increasing number?

   (A) 19 **(B)** 20 **(C)** 21 **(D)** 22 **(E)** 23

   proposed by: Christopher Li

   Solution: The largest such number is 2379 since having more than 4 digits would force either a repeat or a number that is not prime. The only other four digit number which satisfies the conditions is 1379, which is smaller. The sum of the digits is the largest prime-increasing number is thus $2 + 3 + 7 + 9 = \boxed{\textbf{(C)}\ 21}$.

7. What is the sum of all positive integers n such that $n!$ ends with exactly 6 zeros? ($n!$ is the product of all positive integers from 1 to n)

   (A) 165 **(B)** 135 **(C)** 140 **(D)** 60 **(E)** 25

   proposed by: Terry Yang

   Solution: Note that the power of 2 will always be larger than that of 5, so we only need to look at the power of 5 in the factorization of $n!$. According to the condition of the problem, we have that the power of 5 in the prime factorization of $n!$ should be 6. From counting the powers of 5 in a factorial, it follows that

   $$25 \leq n < 30.$$

   Since anything smaller would have a power of 5 which is too small, and any larger number would have a power of 5 which is too large. Therefore, the sum of all positive integers n such that $n!$ ends with exactly 5 zeros is

   $$25 + 26 + 27 + 28 + 29 = \boxed{\textbf{(B) } 135}.$$

8. Points A and B lie on circle ω with radius 2 such that $\overline{AB} = 2\sqrt{3}$. If point C is chosen uniformly at random on ω, what is the probability that $\angle ACB = 60°$?

   (A) 0 **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2}{3}$ **(E)** 1

   proposed by: Daniel Chen

   Solution: Let O be the center of ω. Since $OA = OB = 2$ and $AB = 2\sqrt{3}$, $\angle AOB = 120°$ by 30-60-90 triangles. Thus by the Inscribed Angle Theorem, $\angle ACB = \frac{1}{2}\angle AOB = 60°$ as long as C does not lie on arc AB. This has probability $\frac{240°}{360°} = \boxed{\textbf{(D)} \frac{2}{3}}$.

9. Distinct positive integers a, b, c, d satisfy $a + b + c + d = 25$. What is the maximum possible value of $ab + cd - ac - bd$?

   (A) 80 **(B)** 85 **(C)** 88 **(D)** 90 **(E)** 110

   proposed by: Oscar Zhou

   Solution 1: $ab + cd - ac - bd = (a - d)(b - c)$. Since we want to maximize this expression, d and c has to be minimized since they are positive integers. Without loss of generality let $(c, d) = (2, 1)$. Then the original expression becomes

$$\begin{aligned}(a-1)(b-2) &= ab - 2a - b + 2 \\ &= ab - 22 + 2 - a \\ &= a(b-1) - 20\end{aligned}$$

Since $a + (b - 1) = (25 - 1) - c - d = 21$, the maximum value of $a(b - 1)$ occurs $(a, b - 1) = (10, 11)$. Then $(a, b, c, d) = (10, 12, 2, 1)$, and that value is $\boxed{\textbf{(D)}\ 90}$.

Solution 2: Factor $ab + cd - ac - bd = (a - d)(b - c)$ as in the first solution. By the AM-GM inequality, we have

$$\begin{aligned}(a-d)+(b-c) &\geq 2\sqrt{(a-d)(b-c)} \\ \iff \left(\frac{a+b-c-d}{2}\right)^2 &\geq (a-d)(b-c) \\ \iff \left(\frac{25-2c-2d}{2}\right)^2 &\geq (a-d)(b-c)\end{aligned}$$

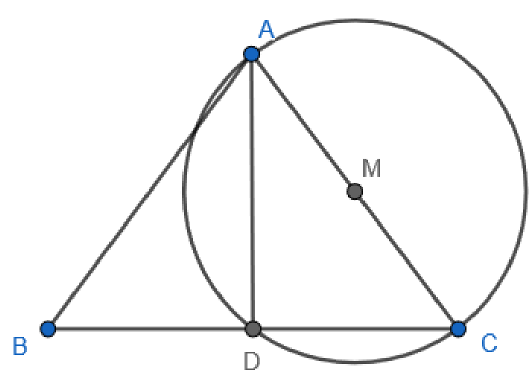
Since c, d are distinct positive integers, the minimal value of $2c+2d$ is $2(1)+2(2) = 6$. Then the maximum value of $(a - d)(b - c)$ is $\frac{361}{4} = 90.25$, but the expression must evaluate to an integer so the maximum possible value is $\boxed{\textbf{(D)}\ 90}$.

10. In $\triangle ABC$, let point D lie on BC such that AD bisects $\angle BAC$. Let M be the midpoint of AC. If $\overline{DM} = 5$, $\overline{AC} = 10$, $\overline{BC} = 12$, find $\overline{AD}$.

(A) $\frac{3}{4}$ **(B)** $\frac{12}{13}$ **(C)** 4 **(D)** 8 **(E)** 13

proposed by: Elaine Li

Solution:



[h]

Since $AM = MC = DM = 5$, point D lies on the circle with diameter AC. This implies that $\angle ADC = 90°$, so $\triangle ADC$ is a right triangle. AD is the angle bisector, so D is the midpoint of BC which makes $\triangle ABC$ isosceles. Then $DC = 6$, and by the Pythagorean Theorem, $AD = \sqrt{AC^2 - DC^2} = \boxed{\textbf{(D) } 8}$.

11. Let $d(n)$ denote the number of positive integer divisors of n. Find the sum of the divisors of the smallest positive integer n such that $d(d(n)) = 6$.

(A) 72 **(B)** 96 **(C)** 168 **(D)** 195 **(E)** 234

proposed by: Daniel Chen

Solution: Let $d(n) = m$. The smallest solution to $d(m) = 6$ is $m = 2^2 \cdot 3 = 12$. Then for $d(n) = 12$, the possible smallest options gives either $2^2 \cdot 3 \cdot 5 = 60$ or $2^3 \cdot 3^2 = 72$ of which 60 is smaller. The sum of the divisors of 60 is $(1 + 2 + 2^2)(1 + 3)(1 + 5) = \boxed{\textbf{(C) } 168}$.

12. What is the shape formed by all possible points (x, y) satisfying the following inequality?

$$|x + y - 1| + |x - y + 1| + |x + y + 1| + |x - y - 1| \leq 4$$

(**A**) Four vertices of a square.
(**B**) A square and its inner region.
(**C**) Four sides of a square.
(**D**) Three vertices of a triangle.
(**E**) Eight points.

proposed by: Terry Yang

Solution: Obviously, $|a| + |b| = |a + b|$ when $ab \geq 0$. According to this inequality, we have that:

$$4 \geq |x + y - 1| + |x - y - 1| + |x + y + 1| + |x - y + 1| = 2|2x - 2| + 2|2x + 2| \geq 4.$$

Thus, we have

$$|2 - 2x| + |2 + 2x| = 4,$$
$$|1 - x - y| + |1 - x + y| = |2 - 2x|,$$
$$|1 + x + y| + |1 + x - y| = |2 + 2x|.$$

We know that the locus are all possible points (x, y) for which

$$\begin{aligned}2-2x &\geq 0,\\ 2+2x &\geq 0,\\ 1-x-y &\geq 0,\\ 1-x+y &\geq 0,\\ 1+x+y &\geq 0,\\ 1+x-y &\geq 0.\end{aligned}$$

Therefore the locus of all possible points (x, y) satisfying the given inequality is $\boxed{\textbf{(B)}\text{ A square and its inner region}}$.

Remark: To solve the problem, it is sufficient to notice the following facts:

- The shape is symmetrical about both $y = x$ and $y = -x$ which eliminates option D.
- All of $(x, y) = (0, 0), (0, 1), (1, 0), (-1, 0), (0, -1)$ work, which eliminate choices A and C.
- $(0, 0)$ is the only point that cannot be reflected about either $y = x$ or $y = -x$, which eliminates option E by parity.

Thus the answer must be $\boxed{\textbf{(B)}\text{ A square and its inner region}}$.

13. The string *abaabaabaaba* consists of characters a and b. A b represents either a plus sign or a minus sign, and a can be any digit from 0 to 9. How many different values could this string evaluate to? For example, one value that it can evaluate to is $0 + 02 - 98 + 20 - 9 = -85$.

    (A) 315 **(B)** 316 **(C)** 622 **(D)** 630 **(E)** 631

    proposed by: Michael Li

    Solution: We can split the a's into tens and ones digits. We have 3 tens digits which sum to -270 at minimum and 270 at most by choosing the sign of the b in front of it. Similarly, we have 5 units digits that sum from -36 to 45. The negative one is -36 because the first a in the string cannot be negative, so the lowest it can be is 0.

    Combining this, the highest possible number achieved is $270 + 45 = 315$, and the lowest possible number is $-270 - 36 = -306$. The middle numbers are achievable by incrementing the digits, so the total possible is $315 + 307 = \boxed{\textbf{(C)}\ 622}$.

14. Participants of a mathematics conference stay in two hotels. Participants staying in the same hotel shook hands with each other exactly once, while participants staying in different hotels did not. The organizers noticed that the total number of handshakes

is coincidentally equal to the product of the number of participants in each hotel. If the total number of participants is greater than 18 but less than 36, what is the total number of participants in the conference?

(A) 20 **(B)** 25 **(C)** 18 **(D)** 30 **(E)** 33

proposed by: Terry Yang

Solution: Let m and n be the number of participants staying in these hotels, then the total number of handshakes is equal to:

$$\binom{m}{2} + \binom{n}{2} = \frac{m(m-1)}{2} + \frac{n(n-1)}{2}.$$

We are given that

$$m(m-1) + n(n-1) = 2mn,$$

Therefore $m + n = (m - n)^2$ by rearranging and we know $18 < m + n < 36$. We obtain that $m + n = 25$ as it is the only perfect square in this range. We can check that $m = 10, n = 15$ satisfy the conditions. Hence, the total number of participants is $\boxed{\textbf{(B) } 25}$.

15. Currently having no money, Bob works for 5 days. On each day, he first receives his salary of 1 gold coin, then he decides whether or not to spend some of the coins that he has saved up to that point. If the coins are indistinguishable, how many ways are there to spend the 5 coins throughout the 5 days such that he has no coins left at the end? For example, a valid way of spending is (1, 0, 2, 0, 2).

    (A) 36 **(B)** 42 **(C)** 84 **(D)** 120 **(E)** 126

    proposed by: Oscar Zhou

    Solution: This problem is equivalent to the number of paths from (0,0) to (5,5) which models the "path" that his total spending takes. There is the added condition that we cannot pass through the line $y = x$ since otherwise Bob would spend more than his savings. But this is just the 5th Catalan number! The answer is $\frac{1}{6}\binom{10}{5} = \boxed{\textbf{(B) } 42}$.

16. Let there be an infinite checkerboard consisting of alternating black and white 1x1 squares. Elaine has a coin of radius $r < \frac{1}{2}$. She tosses the coin onto the checkerboard at random, and wins if it does not touch any black square. The maximum value of r such that the probability of Elaine winning is at least $\frac{1}{3}$ can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, c are integers and b is square-free. Find $a + b + c$.

(A) -3 **(B)** 5 **(C)** 6 **(D)** 13 **(E)** 15

proposed by: Shanna Xiao

Solution: The only way for the coin to not touch a black square is to land in a square of side length $1 - 2r$ inside any white square. There is a $\frac{1}{2}$ chance of it landing in the white square, giving $\frac{(1-2r)^2}{2} > \frac{1}{3}$. Arranging gives the quadratic $12r^2 - 12r + 1 > 0$, and solving yields $r = \frac{3-\sqrt{6}}{6}$. So $a = 3, b = 6, c = 6$ and $a + b + c = \boxed{\textbf{(E) } 15}$.

17. A function $f(x)$ satisfies $f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$. What is a possible value of $f(0)$?

(A) -2 **(B)** -1 **(C)** 2 **(D)** $3 + 2\sqrt{2}$ **(E)** 4

proposed by: Michael Li

Solution: Substitute $x = 0$. We have $f(f(0)) = 0$. Next, we substitute $x = f(0)$. We get:

$$f(f(f(0))) = (f(0))^4 - 6(f(0))^3 + 10(f(0))^2 - 3f(0).$$

This gives

$$f(0) = (f(0))^4 - 6(f(0))^3 + 10(f(0))^2 - 3f(0) \implies f(0)(f(0)-2)(f(0)-2-\sqrt{2})(f(0)-2+\sqrt{2}) = 0$$

Therefore, the possible values of $f(0)$ are 0, 2, $2 \pm \sqrt{2}$. The only available choice is $\boxed{\textbf{(C) } 2}$.

18. Two balls with diameters 5 and d, respectively, can be placed inside a rectangular box with dimensions $5 \times 5 \times 7$. If the maximum possible value of d can be written as $a - \sqrt{b}$ for integers a and b where b is not a perfect square, compute $a + b$.

(A) 2 **(B)** 43 **(C)** 59 **(D)** 76 **(E)** 103

proposed by: Daniel Chen

Solution: To achieve the maximum radius, the second ball must be tangent to the three walls of the box as well as the first ball. By using the Pythagorean theorem in 3 dimensions to calculate the distance between the centers of the spheres, we have

$$\left(\frac{9-d}{2}\right)^2 + \left(\frac{5-d}{2}\right)^2 + \left(\frac{5-d}{2}\right)^2 = \left(\frac{5+d}{2}\right)^2.$$

Rearranging gives $d^2 - 24d + 53 = 0$. Using the quadratic formula, we see that $d = 12 \pm \sqrt{91}$. $12 + \sqrt{91}$ is clearly too large, so $d = 12 - \sqrt{91}$. $a + b = \boxed{\textbf{(E) } 103}$.

19. What is the product of all possible real numbers x which satisfy the equation

$$\log_x(2) + \left(\log_2\left(\frac{x}{4}\right)\right)^3 = \log_{16}(2x^{11}) - 3?$$

(A) 1 **(B)** 2 **(C)** 32 **(D)** $32\sqrt{2}$ **(E)** 64

proposed by: Pavel MacKenzie

Solution: For ease of notation, write $y = \log_2(x)$. We can reduce the equation to a polynomial in y:

$$\begin{aligned} & \log_x(2) + \left(\log_2\left(\frac{x}{4}\right)\right)^3 = \log_{16}(2x^{11}) - 3 \\ \iff \quad & \frac{1}{y} + (y-2)^3 = \frac{1}{4}(1+11y) - 3 \\ \iff \quad & \frac{1}{y}(y^4 - 6y^3 + \frac{37}{4}y^2 - \frac{21}{4}y + 1) = 0 \\ \iff \quad & \frac{1}{y}\left(y - \frac{1}{2}\right)^2 (y-1)(y-4) = 0 \end{aligned}$$

Thus the possible values of y are $\frac{1}{2}, 1$, and 4. This gives solutions $x = 2^{\frac{1}{2}}, 2^1$, and 2^4, multiplying to $2^{\frac{11}{2}}$ or $\boxed{\textbf{(D) } 32\sqrt{2}}$.

20. Let n be the number of nonreal complex numbers z such that $|z|$ and $|z - 2024|$ are both integers less than 2024. What are the rightmost 2 digits of n?

(A) 06 **(B)** 29 **(C)** 59 **(D)** 76 **(E)** 00

proposed by: Pavel MacKenzie

Solution: What the problem is really asking is for is the number of non-degenerate triangles with integer side lengths such that one of the sides has length 2024 and the other two sides have length less than 2024. This is because if we take any one of these triangles, we can simply translate and rotate it into a position that satisfies the conditions of the problem.

So, envision the two equations as sets of concentric circles around the points $A = (0, 0)$ and $B = (2024, 0)$ with radii $0, 1, 2 \ldots 2023$. The circle with radius 0 at B does not intersect any other circle and the circle with radius 1 intersects with a circle from A only at 1 real point $(2023, 0)$. The circle from B with radius 2 intersects the circle from A with radius 2023 twice and the circle from A with radius 2022 at one real point. Extending this to the circle with radius r, we can see that it has $2r-2$ nonreal intersections. Tallying these solutions up over all possible radii, we get

$$\sum_{i=1}^{2024} 2i - 2 = 2 \cdot \frac{2024 \cdot 2025}{2} - 2 \cdot 2024$$
$$= 4090506$$

so the answer is $\boxed{\textbf{(A) } 06}$.

21. Let $a_0, a_1, \dots a_{2024}$ be a sequence of numbers with $a_0 = 1$ and $a_1 = 500$. If the equation

$$(a_n)(a_{n-2})^4 = (a_{n-1})^5 - (a_{n-1}a_{n-2})^4$$

holds for all $n \geq 2$, compute a_{2024}.

(A) -2024 **(B)** 0 **(C)** 2024 **(D)** 2024^3 **(E)** 2^{2024}

proposed by: Elaine Li

Solution: Dividing both sides by $a_{n-1}^4 a_{n-2}^4$, we get

$$\frac{a_n}{(a_{n-1})^4} = \frac{a_{n-1}}{(a_{n-2})^4} - 1.$$

Let $b_n = \frac{a_n}{(a_{n-1})^4}$. Substituting this in, we get $b_n = b_{n-1} - 1$. Since $a_1 = 500$ and $a_0 = 1$, $b_1 = \frac{a_1}{(a_0)^4} = 500$. Thus, $b_2 = 499$, $b_3 = 498$, ..., $b_{501} = 0 = \frac{a_{501}}{(a_{500})^4}$, so $a_{501} = 0$.
Rearranging the original equation, we also see that

$$a_n = (a_{n-1})^4 \times (\text{other stuff})$$

Thus, we see that if $n \geq 501$, $a_n = 0$ which implies $a_{2024} = 0$. The answer is then $\boxed{\textbf{(B) } 0}$.

22. In quadrilateral $ABCD$, $AB = 7$, $BC = 5$, and $CD = 6$. Points E, F are the midpoints of AB and CD respectively, and M, N are the midpoints of AC and BD respectively. If we define $S = EM + MN + NF$, then there exists a real number X such that $S < X$ for all possible quadrilaterals $ABCD$. If the minimum possible value of X can be written as $\frac{a}{b}$ for integers a and b, what is $a + b$?

(A) 6 **(B)** 7 **(C)** 23 **(D)** 25 **(E)** 27

proposed by: Shanna Xiao

Solution: Since $AE = EB$ and $AM = MC$, we have $EM \parallel BC$. Similar triangles yields $EM = \frac{1}{2}$ and $BC = \frac{5}{2}$. Similarly, $NF = \frac{1}{2}$, $BC = \frac{5}{2}$. Let G be midpoint of AD. Using the same reasoning as before with midpoints, we get $MG = 3$, $NG = \frac{7}{2}$. By the triangle inequality,

$$MG + NG = 3 + \frac{7}{2} = \frac{13}{2} > MN.$$

Therefore $S = EM + MN + NF < 5 + \frac{13}{2} = \frac{23}{2}$. $a + b = \boxed{\textbf{(D) } 25}$.

23. If a and b are positive integers satisfying

$$7 \gcd(a^2 + b, a + b^2) = \operatorname{lcm}(a, b),$$

compute the smallest possible value of $a + b$.

(A) 16 **(B)** 23 **(C)** 28 **(D)** 32 **(E)** 48

proposed by: Daniel Chen

Solution: Notice that $a \mid \operatorname{lcm}(a, b)$ and $\gcd(a^2 + b, a + b^2) \mid a^2 + b$. Then $a \mid 7(a^2 + b)$, so $a \mid 7b$. By the same reasoning, we have that $b \mid 7a$. Combining these two facts, we find that there are only two possible cases.

Case 1: $a = b$.
Then $7(a^2 + a) = a$, and rearranging gives $a(7a + 6) = 0$. The possible solutions for a are not positive integers, so there are no possible solutions in this case.

Case 2 $b = 7a$ (also $a = 7b$ but they are the same thing)
Plugging this in gives the following:

$$\begin{aligned} & 7 \gcd(a^2 + 7a, a + 49a^2) = 7a \\ \iff \quad & \gcd(a + 7, 1 + 49a) = 1 \\ \iff \quad & \gcd(a + 7, -342) = 1 \end{aligned}$$

where we used the Euclidean Algorithm. The smallest possible value of a that satisfies this is $a = 4$, which gives $b = 28$. The answer is $a + b = \boxed{\textbf{(D) } 32}$.

24. Turbo the Snail starts in the middle cell of a 5×5 grid. If on every move he travels to an orthogonally adjacent cell (a cell that shares a side with his current cell) with uniform probability, what is the expected number of moves that he will take before arriving at the center once again?

(A) 13 **(B)** 14 **(C)** $\frac{27}{2}$ **(D)** 19 **(E)** 20

proposed by: Daniel Chen

Solution: By symmetry, we can classify all the cells in the grid into five categories:

E	D	C	D	E
D	B	A	B	D
C	A		A	C
D	B	A	B	D
E	D	C	D	E

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Using states, we can write the following system of equations:

$$
\begin{aligned}
E[A] &= \frac{1}{4} + \frac{1}{2}(E[B]+1) + \frac{1}{4}(E[C]+1) \\
E[B] &= \frac{1}{2}(E[A]+1) + \frac{1}{2}(E[D]+1) \\
E[C] &= \frac{1}{3}(E[A]+1) + \frac{2}{3}(E[D]+1) \\
E[D] &= \frac{1}{3}(E[B]+1) + \frac{1}{3}(E[C]+1) + \frac{1}{3}(E[E]+1) \\
E[E] &= E[D]+1
\end{aligned}
$$

For simplicity, define $A' = E[A]+1$, $B' = E[B]+1$, $C' = E[C]+1$, and $D' = E[D]+1$. Then the system becomes

$$
\begin{aligned}
A' &= \frac{5}{4} + \frac{1}{2}B' + \frac{1}{4}C' \\
B' &= 1 + \frac{1}{2}A' + \frac{1}{2}D' \\
C' &= 1 + \frac{1}{3}A' + \frac{2}{3}D' \\
D' &= \frac{4}{3} + \frac{1}{3}B' + \frac{1}{3}C' + \frac{1}{3}D'
\end{aligned}
$$

Plugging in the values for B' and C' into the last equation and solving, we get $D' = \frac{36}{5} + A'$. Substituting everything into the first equation, we get $A' = 20$. Since we needed one move to get to A at the beginning, this is precisely the answer, which is $\boxed{\textbf{(E) } 20}$.

25. A game is played on a regular hexagon with the following rules:

- Each vertex starts with a value of 14.
- On each move, you can decrease the value of three adjacent vertices each by 1.
- No vertex is allowed to have a value less than 0 at any point.

Let n be the number of ways to make a series of moves such that all vertices end with a value of 0. Compute the remainder when n is divided by 7.

(A) 0 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** 6

proposed by: Daniel Chen

Solution: Let a, b, c, d, e, f represent the number of moves made centered around each vertex. Since all vertices are 0 at the end of the game, we must have $a + b + c = 14$ and $b + c + d = 14$, meaning that we must have $a = d$. Similarly, we have $b = e$ and $c = f$. For a fixed value of a, b, c, we can compute the number of ways to arrange the 28 total moves as

$$\frac{28!}{a!b!c!d!e!f!} = \frac{28!}{(a!b!c!)^2}.$$

Examining this expression modulo 7, we see that this expression is equivalent to 0 unless (a, b, c) is a permutation of $(0, 7, 7)$ or $(0, 0, 14)$. Thus these two scenarios are all we need to consider as all the other ones do not contribute to the answer. For $(0, 7, 7)$, we have

$$\begin{aligned} \frac{28!}{(7!)^4} &\equiv \frac{(-1)^4 \cdot 4!}{(-1)^4} \pmod 7 \\ &\equiv 3 \pmod 7 \end{aligned}$$

where we applied Wilson's Theorem to get $(p - 1)! \equiv -1 \pmod p$. Similarly, for the $(0, 0, 14)$ case, we have

$$\begin{aligned} \frac{28!}{(14!)^2} &\equiv \frac{(-1)^4 \cdot 4!}{(-1)^4 \cdot 2^2} \pmod 7 \\ &\equiv 6 \pmod 7. \end{aligned}$$

There are three possible permutations of each case, which gives a final answer of $3(3 + 6) \equiv 6 \pmod 7$ or $\boxed{\textbf{(E) } 6}$.