Tesseract 2024 Official Solutions

Ontario Competitive Mathematics Committee

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Problems

1. A person's birth score is given by multiplying the month and date of their birthday together. For example, a person born on February 9th would have a birth score of $2 \times 9 = 18$.

   Proposed by: Daniel Chen

   (a) What is the birth score of a person born on December 25?

   Solution: The birth score is equal to $12 \times 25 = \boxed{300}$.

   (b) Naomi and Carl both have a birth score of 252, with Naomi's birthday being before Carl's. How many days later is Carl's birthday compared to Naomi's?

   Solution: The prime factorization of 252 is $2^2 \times 3^2 \times 7$. Since the month can be at most 12 and the day can be at most 31, the only possible dates are September 28 and December 21. December 21 is $3 + 31 + 30 + 20 = \boxed{84}$ days after September 28.

2. Ms. Yang bought 500 candies from the supermarket using one dollar, five dollar and ten dollar bills, with one candy costing exactly 1 dollar.

   Proposed by: Michael Li

   (a) If she uses exactly three 10 dollar bills but any number of 5 dollar and 1 dollar bills, how many ways can Ms. Yang pay for the candy?

   Solution: There is \$30 "used" by the 3 ten dollar bills. We divide the remaining \$470 among five and one dollar bills. Consider any number of valid \$5 bills to use, and for the rest, we fill it with one dollar bills. The minimum number of \$5 is 0, and the maximum is $\frac{470}{5} = 94$ five dollar bills. We can choose any number from 0 to 94, for a total of $\boxed{95}$ combinations.

   (b) How many possible ways can she pay for the candy if she can use any number of each of the bills?

   Solution: Proceed with casework based on the number of \$10 bills, then follow the same logic from (a).

   Case 1,0 ten dollar bills:
   Minimum number of \$5 bills: 0
   Maximum number of \$5 bills: 100
   This gives a total of 101 choices

   Case 2, 1 ten dollar bills:
   Minimum number of \$5 bills: 0
   Maximum number of \$5 bills: 98
   For a total of 99 choices

   Case n, $n \leq 50$, with n \$1 dollar bills:
   Minimum number of \$5 bills: 0
   Maximum number of \$5 bills: $\frac{500-10n}{5}$
   For a total of $100 - 2n + 1 = 101 - 2n$ choices.

   Now we sum all of these cases.
   Note that n can range from 0 to 50, so the total number of choices is...

$$
\begin{aligned}
&(101 - 2 \cdot 0) + (101 - 2 \cdot 1) + \cdots + (101 - 2 \cdot 50) \\
&= 51 \cdot 101 - 2(1 + 2 + \cdots + 50) \\
&= \boxed{2601}
\end{aligned}
$$

3. Tommy and Freddy are learning about function transformations in class. Getting extremely bored, Tommy decided to doodle functions on his notebook. He doodles the graph of $y = f(x)$ on one page, and doodles another graph of $y = g(x)$ on another page by reflecting $f(x)$ across the y-axis and translating it to the right by 4 units. Interestingly, he finds that $f(x)+g(x) = 10$ for all real numbers x.

   Proposed by: Oscar Zhou

   (a) Tommy discovered that by turning $f(x)$ about point (m, n) by 180 degrees, it turns back into $f(x)$ again! What is $m + n$? (This point works for all possible functions $f(x)$)

   Solution: The function $g(x)$ is obtained by applying the said function transformations on $y = f(x)$, which yields $g(x) = f(4 - x)$. Since $f(x) + f(4-x) = 10, f(x)$ is symmetrical about point $(2, 5)$. This is because every pair of points that has x-coordinates adding up to 4 has y-coordinates adding up to 10. Point symmetry means that $m = 2$ and $n = 5$, and $m + n = \boxed{7}$.

   (b) Freddy, being the excellent student that he is, listened carefully in class (be like Freddy!). He grabbed Tommy's page with the graph $y = f(x)$ on it and started practicing. He drew the graph $y = 2024(3x - 6)^{2025} + 5$ on top of $y = f(x)$. If there are exactly 2023 intersections between the two graphs, what is the sum of all x-coordinates and y-coordinates of those intersection points?

   Solution: Let $h(x) = 2024(3x-6)^{2025}+5$. Note that the graph of $y = x^{2025}$ has point symmetry around $(0, 0)$, and $h(x)$ is a transformation of $y = x^{2025}$ such that $h(x)$ has point symmetry around $(2, 5)$. Since $f(x)$ and $h(x)$ both have point symmetry around $(2, 5)$, their intersections would also be symmetrical about point $(2, 5)$. This means that the x-coordinates and y-coordinates of 1011 pairs of intersections add up to 4 and 10, respectively, while the last intersection point is $(2, 5)$, which gives the answer $4 \times 1011 + 10 \times 1011 + 2 + 5 = \boxed{14161}$.

4. Let ABC be a triangle with $AB = 13, BC = 14, AC = 15$. Let D be on BC such that AD bisects $\angle BAC$, and E on AC with $DE \perp AC$.

Proposed by: Shanna Xiao

(a) Find the length of AD.

Solution: By Angle Bisector Theorem, $BD = \frac{AB}{AB+AC} = \frac{13}{2}$. Which means that $CD = \frac{15}{2}$. By Stewart's Theorem:

$$(BD)(BC)(DC) + (AD)^2(BC) = (AC)^2(BD) + (AB)^2(DC)$$

Substitute in values for BD, CD, AB, AC, BC to solve for AD, $AD = \boxed{\frac{3\sqrt{65}}{2}}$

(b) Extend ED past D to intersect line AB at X. Find the length of BX.
Solution: Extend ED past E to F where AF is parallel to BC. Then $\triangle AFX \sim \triangle BDX$.
Construct H on BC such that $AH \perp BC$. By Heron's formula, the area of the triangle is 84. Solving for AH, $AH = \frac{2 \cdot 84}{BC} = 12$. By Pythagorean Theorem, $BH = 5$ and $CH = 9$.
Then $\cos \angle ECD = \frac{HC}{AC} = \frac{9}{15} = \frac{3}{5}$, and,

$$EC = DC \cdot \cos \angle ECD = \frac{15}{2} \cdot \frac{3}{5} = \frac{9}{2}.$$

Which then gives that $AE = \frac{21}{2}$. Since $AF \parallel BC$, we also have $\triangle DCE \sim \triangle FAE$ with a ratio of 3: 7. Therefore, $AF = \frac{7}{3} \cdot CD = \frac{35}{2}$.
Now we use $\triangle AFX \sim \triangle BDX$,

$$\begin{aligned}
\frac{BX}{AX} &= \frac{BD}{AF} \\
\frac{BX}{BX+13} &= \frac{\frac{13}{2}}{\frac{35}{2}} \\
\frac{BX}{BX+13} &= \frac{13}{35} \\
35BX &= 13BX + 169 \\
22BX &= 169 \\
BX &= \boxed{\frac{169}{22}}.
\end{aligned}$$

5. Shiro is playing a game where she has a score of x which is initially equal to 1. For each positive integer $2, 3, \ldots, 100$ (in that order), Shiro can choose one of the following actions:

   - Add the number to x.
   - Divide x by the number.

   However, Shiro must never let x become less than 1 after any action.

   Proposed by: Daniel Chen

   (a) Compute the minimum possible value of x after using all 99 numbers.

   Solution:
   The minimum value of x after all 99 operations is

$$1 + \frac{1}{2^{48} \times 50!}.$$

This value is indeed achievable by taking $\frac{1+2+3}{4} = \frac{3}{2} = 1 + \frac{1}{2}$, then alternating between adding and dividing:

$$\frac{x + 2k - 1}{2k} = 1 + \frac{x-1}{2k}$$

So our final value is

$$1 + \frac{1}{2} \times \frac{1}{6} \times \frac{1}{8} \times \frac{1}{10} \times \ldots \times \frac{1}{100} = 1 + \frac{1}{2^{48} \times 50!}$$

Now we prove that this is indeed the minimum. Let m_i be the minimum possible value of x after using the number i. By definition,

$$m_i = \min\left(m_{i-2} + 2i - 1, \frac{m_{i-2} + i - 1}{i}, \frac{m_{i-2}}{i(i-1)}, \frac{m_{i-2}}{i-1} + i\right).$$

Of the 4 options, $\frac{m_{i-2}}{i(i-1)}$ is the smallest, but

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} < (n+1)(n+2),$$

so x would become less than 1, absurd. Therefore, the next smallest option is $\frac{m_{i-2}+i-1}{i}$. Proceeding by induction and taking this optimal option at each step, we eventually arrive at m_4, which we can manually check has a minimum of $\frac{3}{2}$. Thus the claimed value is indeed the minimum.

Remark: Since part A is short answer, simply realizing the construction would yield full marks.

(b) Find the number of possible values of $\lfloor x \rfloor$ after using all 99 numbers.

Solution:

I claim that if Shiro uses the numbers $2, 3, \ldots, n-1$ in some way and chooses to divide the number $n \geq 3$, the resulting x immediately after this action must satisfy $\lfloor x \rfloor \in [1, \lfloor \frac{n-1}{2} \rfloor]$, and $\lfloor x \rfloor$ can achieve any integer in that range. $\lfloor x \rfloor \geq 1$ is obvious. The maximum is achieveable by simply adding up all the numbers before n before dividing by n,

$$x = \frac{1 + 2 + \ldots + n - 1}{n} = \frac{n-1}{2}$$

This is clearly the greatest x achievable. For $n = 3, 4$, we are done. For $n \geq 5$, consider the following process: choose some integer $3 \leq k \leq n-2$. Add all numbers $2, 3, \ldots, k-1$, then divide by k, then add the rest of the numbers $k+1, \ldots, n-1$, and of course, divide by n at the end. At no point during the process will x be below 1, and the resulting x will be

$$\begin{aligned} x_k &= \frac{\frac{1}{k} \sum_{i=1}^{k-1} i + \sum_{i=k+1}^{n-1} i}{n} \\ &= \frac{n-1}{2} - \frac{k^2+1}{2n}, \end{aligned}$$

which decreases as k increases. If we pick $k = 3$, then $x_3 = \frac{n-1}{2} - \frac{5}{n}$. Note that $\lfloor x_3 \rfloor \geq \lfloor \frac{n-1}{2} \rfloor - 1$. If we pick $k = n-2$, then $x_{n-2} = \frac{3}{2} - \frac{5}{2n}$, and $\lfloor x_{n-2} \rfloor = 1$. As we decrease from x_3 to x_4 all the way to x_{n-2}, note that at every step we decrease by

$$x_k - x_{k+1} = \frac{(k+1)^2 - k^2}{2n} = \frac{2k+1}{2n},$$

which is less than 1. Hence, the sequence $\lfloor x_k \rfloor$ never "skips" an integer and can achieve all integers in $[1, \lfloor \frac{n-1}{2} \rfloor]$, so we are done.

Let the last division across the 99 actions be at the number $y \geq 3$. We will then add each of $y+1, y+2, \ldots, 100$ to x. Using our previous result, this creates $\left\lfloor \frac{y-1}{2} \right\rfloor$ possible values of $\lfloor x \rfloor$ in the end. Additionally, no two distinct selections for the final division $y_1 > y_2$ can yield identical $\lfloor x_1 \rfloor, \lfloor x_2 \rfloor$. This is because

$$x_1 \geq \left(1 + (y_1 + 1) + \sum_{i=y_1+2}^{100} i\right) > \left(1 + \left\lfloor \frac{y_2 - 1}{2} \right\rfloor + \sum_{i=y_2+1}^{100} i\right) \geq 1 + x_2$$

Thus over all y, the total possible number of values of $\lfloor x \rfloor$ is

$$\begin{aligned} \sum_{y=3}^{100} \left\lfloor \frac{y-1}{2} \right\rfloor &= 2 \sum_{y=1}^{49} y \\ &= 2450 \end{aligned}$$

There is also the case where we do not divide at all, giving a distinct additional $\lfloor x \rfloor$ for a total of $\boxed{2451}$ possible values.