1. Atticus has a bag which contains 3 red balls, 2 blue balls, 5 green balls and no balls of any other colour.

   (a) If Atticus randomly removes a ball from the bag, what is the probability that it is blue?

   (b) Atticus adds another n red balls to the bag, and now the probability that a randomly removed ball is red is $\frac{1}{2}$. What is the value of n?

proposed by: Daniel Chen

Solution:
(a) There are a total of 10 balls, of which 2 are blue. The answer is thus $\frac{2}{10}$ which simplifies to $\boxed{\dfrac{1}{5}}$.

(b) For part b, adding n red balls brings the total number of balls to $10 + n$ and the number of red balls to $3 + n$. We then have the equation

$$\frac{3+n}{10+n} = \frac{1}{2}.$$

Rearranging and solving the equation, we get $n = \boxed{4}$.

2. (a) Find the unique real number x such that $x, \frac{x}{2}, x+1$ forms an arithmetic sequence in that order.

   (b) Find the unique positive real number x such that $x, 3x+1, x^2$ forms an arithmetic sequence in that order.

   (c) Find the unique positive real number x such that x, $\lfloor x \rfloor$, and $x - \lfloor x \rfloor$ forms a arithmetic sequence in that order. (The floor function $\lfloor x \rfloor$ evaluates to the largest integer less than or equal to x)

proposed by: Michael Li

Solution:

(a) By the definition of an arithmetic progression, we must have

$$2 \cdot \frac{x}{2} = x + (x+1) \implies x + 1 = 0 \implies x = -1$$

Thus, our only solution is $x = \boxed{-1}$.

(b) Similarly to part (a), we get an equation:

$$x + x^2 = 2(3x+1) \iff x^2 - 5x - 2 = 0.$$

Using the quadratic equation, we get $x = \frac{5 \pm \sqrt{33}}{2}$. The only positive solution is $x = \boxed{\dfrac{5+\sqrt{33}}{2}}$.

(c) Notice that when $0 < x < 1$, $x = x - \lfloor x \rfloor$. But then $\lfloor x \rfloor = 0$, so there is no solution. We conclude that $x \geq 1$. If this is the case, we know that $x - \lfloor x \rfloor \leq \lfloor x \rfloor \leq x$. So, because we have an arithmetic progression, we have

$$\frac{x - \lfloor x \rfloor + x}{2} = \lfloor x \rfloor \implies 2x = 3\lfloor x \rfloor \implies x = \frac{3}{2}\lfloor x \rfloor$$

Now, we test a few values for $\lfloor x \rfloor$ and find the corresponding x value. If it works, great! We test out small values of $\lfloor x \rfloor$, since with larger values, this equation is obviously unsolvable. Let's test out $\lfloor x \rfloor = 1, 2, 3, 4$. Of those, only $\lfloor x \rfloor = 1$ works, corresponding to an answer of $x = 1.5$ or $\boxed{\dfrac{3}{2}}$.

3. For a positive integer n, define $S(n)$ to be the sum of the digits of the base-10 representation of n. For example, $S(6150) = 6 + 1 + 5 + 0 = 12$.

   (a) Compute the number of two digit positive integers n such that $S(n) = S(2n)$.

   (b) Find the smallest positive integer n such that $S(n)$ and $S(n + 11)$ are both divisible by 13.

proposed by: Daniel Chen, Christopher Li

Solution:

(a) Let $n = 10a + b$ where $0 \leq a, b \leq 9$ and $a \neq 0$. Note that $S(2n) = S(20a + 2b) = S(2a) + S(2b)$ since the units digits of $2a$ is never 9 so it is impossible for the tens digit of $2b$ to affect the total digit sum by reaching 10. Let $F(n) = S(2n) - S(n)$. We can write out all values of $F(n)$ for $0 \leq n \leq 9$:

$F(0) = 0$	$F(1) = 1$	$F(2) = 2$	$F(3) = 3$	$F(4) = 4$
$F(5) = -4$	$F(6) = -3$	$F(7) = -2$	$F(8) = -1$	$F(9) = 0$

We know that $S(2n) - S(n) = (S(2a) + S(2b)) - (S(a) + S(b)) = F(a) + F(b)$. In order for this to be 0, it must be the case that $a = 9 - b$ according to the calculations above. This gives $\boxed{9}$ possible pairs as 09 is not a valid two-digit number.

(b) $S(n + 11) = S(n) + S(11) - 9k = S(n) + 2 - 9k$, where k is the number of carry-overs in the addition of n and 11. So $9k \equiv 2 \pmod{13}$, which gives $k \equiv 6 \pmod{13}$. The minimum possible value is $k = 6$. Note that all digits that are carried over in the addition $n + 11$ must be at the end of n. So either n ends in 999999 or 999989.

In the first case, the digits in front must be $11 \pmod{13}$. The minimum is 3 and 8, to ensure no extra carry-overs. So $n = 38999999$ is the smallest for this case.

In the second case, the digits in front must be $12 \pmod{13}$. The minimum is 4 and 8, which creates 48999989.

Thus, $n = \boxed{38999999}$ is the smallest.

4. A game called "24 point" is played by giving the player four positive integers and asking them to make the number 24 by only using addition, subtraction, multiplication, division, and parenthesis. For example, the winning operation for 4, 4, 10, 10 is $(10 \times 10 - 4)/4 = 24$. Note that there may be multiple winning operations for a given set of four positive integers.

   (a) Determine a winning operation for 2, 5, 5, 10.

   (b) Given any four positive integers, prove that by only using addition, subtraction, multiplication, division, and parenthesis, a multiple of 24 can be always be made.

proposed by: Oscar Zhou

Solution:
(a) $(5 - 2/10) \times 5 = 24$ works.

(b) By prime factorizing 24, we notice that a positive integer is a multiple of 24 if and only if it is both a multiple of 3 and a multiple of 8. We will prove that out of the four positive integers, we can always find two that make a multiple of 8, and the remaining two can always make a multiple of 3. Then, by multiplying the two resulting numbers together we achieve a multiple of 24.

First, we construct the multiple of 8.

- If any of the four numbers are already a multiple of 8, then simply pick any other number to multiply it with.
- If any two of the numbers are congruent mod 8 or any two are additive inverses mod 8 (eg. 1 and 7), they can make a multiple of 8 via subtraction or addition, respectively.
- Otherwise, the four integers must contain one of each of $\pm 1, \pm 2, \pm 3$, and 4 modulo 8. In that case, multiplying the integer that is ± 2 mod 8 and the integer that is 4 mod 8 achieves a multiple of 8.

Next, we will prove that any two positive integers can make a multiple of 3.

- If any of the two numbers is a multiple of 3, multiply them.
- If they are equivalent modulo 3, subtract them to get a multiple of 3.
- Otherwise, the numbers must be additive inverses mod 3, so add them.

Therefore, the claim is proven.

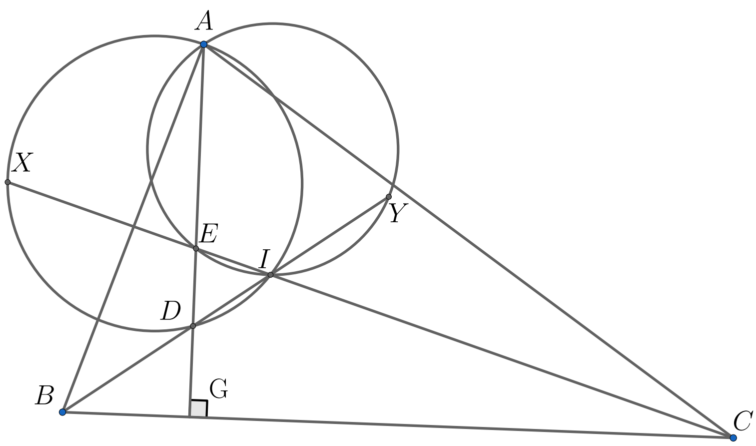
5. In non-isosceles acute triangle ABC, I is the incenter. The angle bisector of B intersects the altitude from A at D. The angle bisector of C intersect the altitude from A to E. The circumcircle of $\triangle AID$ intersects line CE at $X \neq I$. The circumcircle of $\triangle AIE$ intersects line BD at $Y \neq I$.

   (a) Prove that XY is parallel to BC.

   (a) Let P be the point such that $PB \parallel AY$ and $PC \parallel AX$. Prove that P is on the circumcircle of $\triangle ABC$.

proposed by: Christopher Li

Solution:



(a) Let G be the foot of the altitude from A to BC. By angle-chasing on cyclic quadrilaterals, we have

$$\angle DXE = \angle DXI = \angle DAI = \angle EAI = \angle EYI = \angle EYD.$$

Also note that

$$\angle XDE = \angle XDA = \angle XIA = 180^\circ - \angle AIC = \frac{1}{2}\angle A + \frac{1}{2}\angle C = 90^\circ - \frac{1}{2}\angle B$$

and $\angle YDE = \angle BDG = 90^\circ - \frac{1}{2}\angle B$. Thus $\angle XDE = \angle YDE$.

By SAA congruence, $\triangle XED \cong \triangle YED$. This implies ED is the perpendicular bisector of XY, so $XY \perp ED$ and $ED \perp BC$, giving the desired result of $XY \parallel BC$.

(b) By angle chasing, we get

$$\angle AIY = 180^\circ - \angle AIB = \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ - \frac{1}{2}\angle C$$

and

$$\angle AYI = 180^\circ - \angle AEI = \angle GEC = 90^\circ - \frac{1}{2}\angle C$$

so $\angle AYI = \angle AIY$. Similarly, $\angle AXI = \angle AIX$, so $AX = AI = AY$. Let P' be the midpoint of minor arc BC of the circumcircle of $\triangle ABC$.

Note that A, I, P' are collinear. By the Incenter-Excenter Lemma, $BP' = IP' = CP'$. By SSA similarity, $\triangle AIY \sim \triangle P'IB$. Clearly, there exists a homothety at I which sends $\triangle AIY$ to $\triangle P'IB$ so $AY \parallel BP'$. Similarly, we can show that $AX \parallel CP'$, and since P is unique, we must have $P' = P$. P' was defined to be on the circumcircle of $\triangle ABC$ so we are done.